Lecture 3 – The Clockwork Universe

Physics for Pedestrians

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1 Dimensional Analysis

Dimensional Analysis is a tool that has widespread use in physics. The purpose of dimensional analysis is to give certain information about the *relations* which hold between the measurable quantities associated with various physical phenomena. This method has the advantage of being rapid: it enables us to dispense with making a complete analysis of the physical system before drawing conclusions. On the other hand, it does not give us as complete information as might be obtained by carrying out a more detailed analysis.

1.1 Dimensions

A physical quantity that may be measured is usually[1] measured with respect to some standard. If the length of an object – say, a table – is to be measured, it is measured using a scale. This scale would say that the table measured, for example,

$$\text{Length of table } = 2 \text{ metres}$$

This is shorthand for saying that if two standard metre scales of 1 metre each were placed of after the other, they would have the same length as that of the table. Thus,

$$\text{Length of table } = \underbrace{2}_{\text{magnitude}} \times \underbrace{1 \text{ metre}}_{\text{unit}}$$

Similarly, if a duration of time is to be measured to be, say, five years,

$$\begin{aligned} \text{Duration } &= \underbrace{1}_{\text{magnitude}} \times \underbrace{1 \text{ year}}_{\text{unit}} \\ &= \underbrace{3 \times 10^7}_{\text{magnitude}} \times \underbrace{1 \text{ second}}_{\text{unit}} \end{aligned}$$

[1]Though not always!

From the above example, it should be clear that the **magnitude** of a physical quantity depends on the unit chosen. If we naively only paid attention to the magnitude, a year might seem like a very small amount of time (just "1" unit), or a very large amount of time (30,000,000 units!).

It is important here to draw a distinction between dimensions (which are physical quantities that may be measured) and units, which are the standards with reference to which they are measured.

For example, length is a dimension, but it may be measured using different units, the yard, the kilometre, the foot, or the light-year.

The physical quantities with dimension that we may measure can be broadly classified into two groups: **primary** and **secondary** quantities. Primary quantities are considered fundamental and irreducible (they cannot be written in terms of other quantities), while secondary quantities may be constructed from combinations primary quantities. The primary quantities that we will refer to through this course are mass (M), length (L), and time (T). An example here would perhaps be enlightening: consider the quantity "speed" or "velocity". It is defined to be the change in distance (Δx undergone by an object in a time interval Δt). i.e.

$$v = \frac{\Delta x}{\Delta t}$$

In terms of dimensions, the numerator of the above equation is a measure of length (and hence has dimension [L]), while the denominator is a measure of time (and hence has dimension [T]). The dimensions of velocity (usually represented by [v]), are given by

$$[v] = LT^{-1}.$$

Question: Show that the dimension of acceleration is

$$[a] = LT^{-2}$$

Question: What are the dimensions of force?

1.2 Principles

Let us begin with two basic principles on which physics is based:

Principle 1: Only magnitudes of quantities of similar dimension can be compared.

This should be quite obvious: it makes no sense to say that an object which is 10 m long is "smaller" than 1000 seconds. Similarly, questions like "Is a kilogram larger than a second", or "How many seconds are there in a metre", or "How long is a gram" don't make any sense.

Principle 2: Physical phenomena and physical laws do not depend on the unit system selected.

This may not be quite so intuitive, but is nevertheless one of the guiding principles of Physics. The units that we have chosen to describe the world, as described in the first lecture, are very personal units, which make sense to us as humans, but would seem very strange to a bug or an elephant.

The very nature of physics as a discipline requires that human beings and our choice of "standards" (such as the kilogram, or the metre, or the second) not be crucial to our description of the universe.[2]

The first principle should make it clear that you cannot add two quantities that have different dimensions (you can't add a kilogram to a metre). The second is far deeper: it means that we could change our units of measurement (say from Imperial to metric, or from human to bug) and our description of Nature would continue to be as valid.

1.3 Constants of Nature

This is particularly important as in Nature we have certain physical constants which have dimension (in other words, they are not merely magnitudes, but are measured in units). Whereas the physical quantity indicated by a physical constant does not depend on the unit system used to express the quantity, the numerical values of dimensional physical constants do depend on choice of unit system.

> The term "physical constant" refers to the physical quantity, and not to the numerical value within any given system of units.

One example is the speed of light

$$c = 3 \times 10^8 \text{ m/s}.$$

As can be seen by the "m/s", the magnitude (3×10^8) of the speed of light depends on the units chosen. If you had chosen to measure it in cm/s, then it would be

$$c = 3 \times 10^{10} \text{ cm/s}.$$

If you had chosen to measure it in cm/hr, it would be

$$c = 1.8 \times 10^{12} \text{ cm/hr} \tag{1}$$

[2] This has sometimes been called the *Copernican Principle*, after Copernicus who argued that the universe – quite literally – did not revolve around us.

1.4 Analysing dimensions

In the above example, the speed of light is measured in units of (say) cm/hr. We can check whether this is **dimensionally consistent**. We know that by definition, speed is a rate of change of distance with respect to time, and so it must have dimension

$$[c] = LT^{-1}$$

and on the right hand side, it is measured in units of cm ([cm] = L) over hour ([hour]=T), thus

$$[cm/hr] = LT^{-1}$$

Thus, looking at the dimensions on either side of Equation (1),

$$\begin{aligned} c &= 1.8 \times 10^{12} \text{ cm/hr} \\ [c] &= [\text{cm/hr}] \\ LT^{-1} &= LT^{-1}. \end{aligned}$$

Notice that in the second line of the above equation the magnitude (1.8×10^{12} suddenly disappears, but we still have an equality sign. The reason for this is because when we use the "[]" notation we are only concerned with the dimensions on either side, which is not affected by the magnitude which is a pure number.

1.5 Method

Let us begin with an example. Suppose you want to find the maximum height h_{max} that a ball thrown upward with some speed u can reach. You spend some time thinking and decide that it could depend on the speed u (if u were greater, so would the height it reaches), the acceleration due to the gravity g (if this experiment were done on the moon, the ball would certainly move higher), and you feel that it would also depend on the mass m of the ball.

h_{max} should thus be some combination of u, g, and m. Since these quantities can't just be added (why not?), we need to combine them in some suitable way to get a quantity with dimension length. So we say

$$\begin{aligned} h_{\text{max}} &\sim \underbrace{u \times u \times u \times \ldots}_{a \text{ times}} \times \underbrace{g \times g \times g \times \ldots}_{b \text{ times}} \times \underbrace{m \times m \times m \times \ldots}_{c \text{ times}} \\ &\sim u^a g^b m^c \end{aligned}$$

It's important to realise that we have used the $\sim$ symbol on purpose. This method of analysis (as we saw in the last section) cannot tell us anything about the dimensionless number in front of these units (like the magnitude 1.8×10^{12} in our last example). Thus, our answer is only very roughly correct, and there will most often be a constant which is larger than 1 in front of it.

Now, we consider only the dimensions on either side.

$$\begin{aligned} &= [u]^a [g]^b [m]^c \\ L &= (LT^{-1})^a (LT^{-2})^b (M)^c \\ L^1 &= L^{a+b} T^{-(a+2b)} M^c \\ M^0 L^1 T^0 &= L^{a+b} T^{-(a+2b)} M^c \end{aligned}$$

The equality sign has replaced $\sim$ since we're only dealing with the dimensions, and this is a pure equality.

In the last step, we compare the dimensions on either side of the equation. The left-hand side, which only has length, has no mass or time, and we represent this by placing mass and time to the power 0.[3] We then compare the powers on either side. Since there is no mass on the left-hand side, this means that on the right-hand side, $c = 0$.

Similarly, since there is no time on the left-hand side, the power of time on the right-hand side should also be zero.

$$\implies -(a + 2b) = 0 \qquad \implies a = -2b$$

And last of all, since there is only one length on the left-hand side, the power of length on the right-hand side must be 1.

$$a + b = 1$$

Question: Show that this implies that

$$h_{\text{max}} \sim \frac{u^2}{g}$$

Doing a more detailed analysis (beyond the scope of this class), you will find that

$$h_{\text{max}} = \frac{1}{2} \frac{u^2}{g}$$

which is not far off!

[3] Since any number to the power 0 is 1, which is dimensionless.

From this, we also reach what might seem to be a strange conclusion: while we had assumed that mass might be a factor, it turns out that the mass does not seem to contribute to the maximum height reached! The reason for this is that **there is no way in this scheme of things, for mass to be placed in the above equation!**

The method used in dimensional analysis is simple. However, using it in more complicated situation becomes an art. Suppose you have a physical situation, and you want to decide quantitatively how some parameter depends on the other parameters of the problem. Here are the steps:

1. **Step 1:** Find the relevant parameters that the problem depends on. In the above example, it would be u, g, and m.
2. **Step 2:** Write the quantity that you are interested in as some product of powers of the other quantities.
3. **Step 3:** Expand these quantities in terms of their fundamental dimensions ($u = LT^{-1}$, $g = LT^{-2}$, etc.) and equate the dimensions on either side, deriving a relation between the powers.
4. **Step 4:** Solve for the powers to get the final relation. Interpret your result (i.e. the mass does not affect the maximum height, etc.)

Obviously, the most important step is the first: finding the quantities that matter, and this is no small task. It requires a slight understanding of physics.

2 Understanding the quantities that matter

In order to understand the parameters that are important, we need a more detailed understanding of physics. Let us begin with a study of *Classical* systems.

2.1 Galileo's Experiments

We will focus on two experiments that Galileo conducted. The first was concerned with the falling of objects, and the second with the natural state of an object.

2.1.1 Free-fall

At the time, it was assumed that objects of different masses took different amounts of time to fall a certain distance. This was in accordance with experiment: a coin and a feather do not take the same amount of time to fall a certain distance.

Galileo performed experiments to test this. He had a set of inclined planes at different angles, and he rolled identical objects down them, with different masses. The result was that at every angle, the two objects reached the bottom at the same time. He carried this out for many different masses and was forced to conclude that the time the object took to roll down an inclined plane was **independent of the mass**!

He then performed a **thought-experiment**: since it was too hard to tell if two objects reached the ground at the same time when they were dropped, he assumed that two objects falling vertically could be thought of as two objects **rolling** down an inclined plane at an angle of 90°! Since the angle did not affect the fact that the masses reached the ground at the same time, he concluded that the time taken by an object to fall a distance was **independent of the mass**.[4]

2.1.2 The principle of inertia

He then carried out experiments on a track which was inclined at different angles on either side (similar to what you saw in class). When an object was released at one end, it was found to speed up until it reached the bottom of the track, and then slow down until it reached the original height again! In other words, the **distance** it covered in the second half of its journey was not important: it was only the **vertical distance** or height that mattered.

While performing this experiment on tracks that were successively steeper, Galileo realised that no matter what happened, the ball would seek its own height. However, in order to do so, the speed of the ball (which was maximum at the bottom) would have to reduce until it was zero at the moment when it attained its original height. However, as the tracks got steeper, the ball would have to go further and further to attain its original height. He then conducted his second thought experiment. He imagined a track which was completely flat (parallel to the floor) and imagined what would happen if he released the ball: on reaching the "bottom" of the track, the ball would have a maximum speed, and it would then slow down until it reached its original height. However, since the track is flat, it can never reach its original height, meaning that it would never slow down but continue at a constant speed forever.

From here, Galileo concluded that the natural state of an object is to be either at rest, or in a state of uniform linear motion (motion in a straight line with a constant velocity.). This is known as the **principle of inertia**.

2.2 Newton's Laws

2.3 The First Law: Reference Frames and Inertia

Newton's first law is basically a restatement of Galileo's principle of inertia. It claims that an object's natural state is one of rest or uniform rectilinear motion.[5] Any deviation from this motion would have to be due to an external influence, known as a **force**.

Using the first law, Newton was able to establish what we now call an **inertial reference frame**. The motion of a body can only be described relative to something else – other bodies, observers, or a set of coordinates in space (and time). For example, imagine your phone lying on a table in a room with the lights out. In order to locate it specifically so that your friend may go in and get it without turning on the lights, you would require to give her three numbers, say, the number of steps she would need to walk directly forward, the number of steps she would have to walk left or right, and the height of the table. These three "coordinates" are sufficient to localise anything in space. Of course, in order for any of this to make sense, you need also to specify that all these numbers are with respect to the starting point, which is the door (this is known as the *origin*).

[4] It depended only on the angle.
[5] That is, motion in a straight line.

We imagine placing ourselves in a reference frame and looking at the motion of different objects around us. (Let us place ourselves in free space to avoid all the pesky effects of gravity.) We may look around and see many different objects moving around, and we sit down and calculate each of their velocities. We find that almost all of them are either standing still or moving at a constant speed, except one little alien with a jetpack, which seems to be increasing it's velocity (i.e. which is *accelerating*).

An inertial reference frame is one in which all accelerations have a clear physical **cause**, which we call a force. Newton's first law only holds in such reference frames.

Armed with this knowledge, you should out to the alien and say "Listen, I have calculated that you're accelerating, and Newton tells me that this means that you feel a force. Can you feel something pushing you?" And the object replies "I do!". Satisfied, you go back to your cataloguing.

The alien, however, pauses for a moment and looks around. In particular, he sees you accelerating *backwards* (since he's accelerating forwards with respect to you, you must be accelerating backwards with respect to him). He then says, "Aha! I see this humanoid accelerating, which means that she must feel a force!". So he shouts out to you[6] "Listen, now *I* have calculated that *you* are accelerating, and according to your man Newton, this should mean that you too feel a force! Can you feel something pushing *you*?", and you reply "No! I feel no forces at all!".

There is thus an asymmetry between these two situations. In one case, the acceleration (that of the alien, with respect to you) was the result of a force, while in the other case the acceleration (that of you, with respect to the alien) was because of the alien's *own* acceleration. Since the alien was in an **accelerating frame of reference**, which is not an inertial frame, he cannot claim that all accelerations are due to forces.

> **Question:** Is our frame of reference – on the surface of the Earth – an inertial frame of reference?

2.4 The Second Law: what do forces do?

Having defined what he meant by a force, Newton goes on to describe what the action of a force means. A force, he claims, changes the natural state that an object is in. Since the natural state is one of uniform velocity, a good guess would be that a force does something to induce a rate of change of velocity.

For simple systems, this just means that the force induces an **acceleration**. This means that the force F and the acceleration of the object a are related by:

$$F \propto a^{7}$$
$$F = m \times a$$

where m is a constant for an object, known as the (inertial) **mass** of the object.

An enlightening way to rewrite the second law is

[6] In space, no one can hear you scream. But let's suspend disbelief for a while.

$$a = \frac{F}{m}.$$

It shows us that the acceleration experienced by an object depends directly on the force applied externally, and depends inverse on the mass of the object. In other words, if you are trying to push two cupboards, one twice as massive as the other, you would require two times the force for the former (as compared to the latter) to get them to move at the same acceleration.[8]

The acceleration is the rate of change of velocity, which is in turn the rate of change of position. Thus,

$$a = \frac{\Delta v}{\Delta t} = \frac{\Delta\left(\frac{\Delta x}{\Delta t}\right)}{\Delta t}$$

Thus, Newton's second law relates the change in velocity (and hence position) of an object due to an external influence (the force). If we know the force, then we could – in principle – solve the above equation exactly to find the position of the particle at every instant of time.

2.5 The Third Law: pairs of forces

In the third law, Newton explained that forces come **in pairs**. Thus, the force that an object 1 exerts on 2 (let's call it F_{12}) is the same as the force that an object 2 exerts on 1 (which we call F_{21}), but in the opposite direction, i.e.

$$F_{12} = -F_{21}.$$

It is important to realise that these two forces to not occur on the same object. (If this had been the case, no object would ever move, as all the forces on it would be perpetually cancelled out!)

2.6 Newton's laws in action: pushing a box on a table

Imagine you had a box on a table that you were trying to push using the force of the muscles in your fingers F_{muscles}, as shown in Figure (1). The object is at rest on a table. The force of gravity is acting downwards on it, and the table exerts a reaction force upwards, known as the normal force. These two forces are equal in magnitude and opposite in direction, so that the box does not accelerate downwards, and remains at rest. This is Newton's First Law in action.

> You might imagine, since the forces are equal in magnitude and opposite in direction, that this is a result of Newton's *Third* Law, but this is not the case, as can be tested by realising that the two forces (Weight and Normal Force) act on the **same object**, i.e. the box. This does not happen in the case where the third law applies, as in that case the force F_{12} acts on the object 2, while the force F_{21} acts on the object 1!

[8] Keep in mind, however, that most of your intuitive understanding of how heavy something is to push is due to the existence of friction on the Earth. This law, however, is also true in free space, where this is no friction. There too, moving a more massive object requires more force.

As you push against the box, it feels a force F_{finger} due to your finger, and it presses back with a force F_{box} which is of the same magnitude. However, this force is no match for F_{muscles}. The forward force from your finger overcomes the frictional force from the table. As can be seen from the figure, there is a net imbalance of forces on the matchbox (since $F_{\text{finger}} >$ Friction) which causes the object to accelerate to the left, as a result of Newton's Second Law.

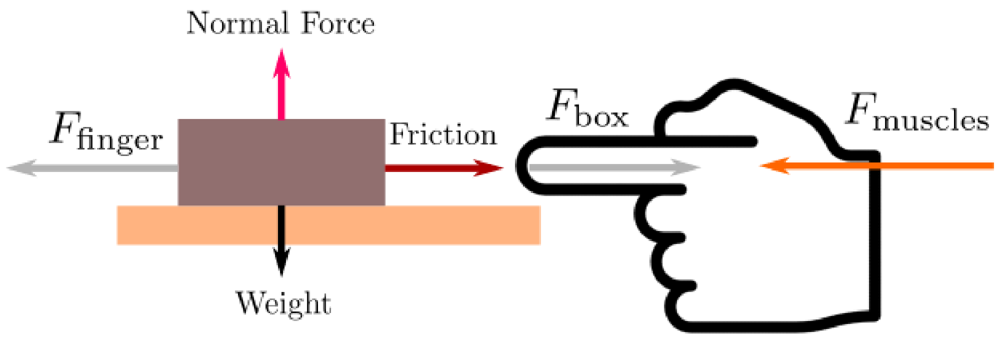

Figure 1: (Works best in colour) Forces acting on a box that's being pushed by your finger. Forces of the same colour are pairs of equal magnitude and opposite direction, from Newton's Third Law. While the Weight and the Normal Force of the ground are *also* equal in magnitude and opposite in direction, they are *not* such a pair, since they act on the same object.

The point of the figure is to make clear that the third law deals with matched pairs of forces that act on **different objects**. Equilibrium from Newton's first or second law deals with the resultant force on a single object.