Very Optional Problem Set – The Special Theory of Relativity

Physics for Pedestrians

1st August, 2019

I have Dr. Bikram Phookun (from St. Stephen's College, Delhi) to thank for a large part of these notes.

1 The Lorentz Transformations

The Lorentz transformations between inertial frames S and S' moving at velocity v with respect to each other along their common x axis are (for $x = x'$ when $t = t' = 0$, i.e. for the origins of S and S' coinciding at the initial moment).

$$\begin{aligned} x' &= \gamma \left(x - v\,t \right) \\ t' &= \gamma \left(t - \frac{v}{c^2}\,x \right) \end{aligned} \tag{1}$$

Notice that the coordinates given above are really the *differences* between the coordinates at a given time and those at the beginning, i.e. between the event (x, t), (x', t') and the event $(0, 0)$ (which has the same coordinates in both frames because of the initial coincidence of their origins). So it is usually more useful to write the transformations in terms of differences between events. We will write $\Delta x = x_B - x_A$ and $\Delta t = t_B - t_A$, where the two events are (x_A, t_A) and (x_B, t_B). Thus we have

$$\begin{aligned} \Delta x' &= \gamma \left(\Delta x - v\,\Delta t \right) \\ \Delta t' &= \gamma \left(\Delta t - \frac{v}{c^2}\,\Delta x \right) \end{aligned} \tag{2}$$

and the inverse transformations

$$\begin{aligned} \Delta x &= \gamma \left(\Delta x' + v\,\Delta t' \right) \\ \Delta t &= \gamma \left(\Delta t' + \frac{v}{c^2}\,\Delta x' \right). \end{aligned} \tag{3}$$

The factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ is always greater than 1 for any $v < c$. (As we shall see, this conditions always holds for material objects.) $1/\gamma$ is therefore always less than 1.

2 Problems

1. Reread the requirements for length measurement given in the notes. Now consider a rod that has length L' in its rest frame S'. In common-sense relativity we found that it had the same length in all other frames as well. Now we will find something different. In the way we have written the transformations above, we can write $L' = \Delta x'$. The measurements of the coordinates of the ends do not need to be simultaneous in S', since the rod is at rest in this frame. However, if its length L is measured in S, with respect to which it is moving, it is necessary to make the measurements simultaneous. Thus we need the Δx corresponding to the $\Delta x' = L'$ under the condition that $\Delta t = 0$.
   (a) Show that under this condition $\Delta x' = \gamma \Delta x$.
   (b) Thus show that $L = L'/\gamma$. Notice that $L < L'$.

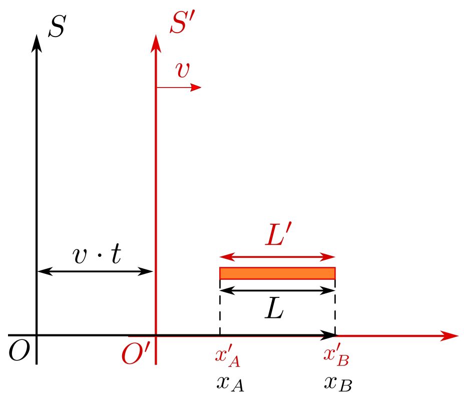

Figure 1: Length measurements

2. Go back to discussion of simultaneity given in section 4.2 in the notes. Let us redo the thought experiment with the Lorentz transformations. In the rest frame of the box. i.e. in S', the time taken for the objects/signals to reach the walls is clearly still $L'/(2\,u')$. So the strikes on the walls are simultaneous in S'.
   (a) Show that the speeds to the right and left are $u_r = (u' + v)/(1 + u'\,v/c^2)$ and $u_l = (u' - v)/(1 - u'\,v/c^2)$.
   (b) Show that $t_r = (L'/(2\,\gamma) + v\,t_r)/u_r$ and $t_l = (L'/(2\,\gamma) - v\,t_l)/u_l$. Solve for t_r and t_l and thus show that $t_r - t_l = \gamma\,L'\,v/c^2$. The strikes on the walls are not simultaneous in S.

   Notice that in $\Delta t = \gamma\,L'\,v/c^2$ the velocity of the objects does not appear.

3. In question above we determined $\Delta t = t_r - t_l$ using two objects/signals travelling at certain undefined velocity. But surely we can equally use light instead of two objects/signals travelling

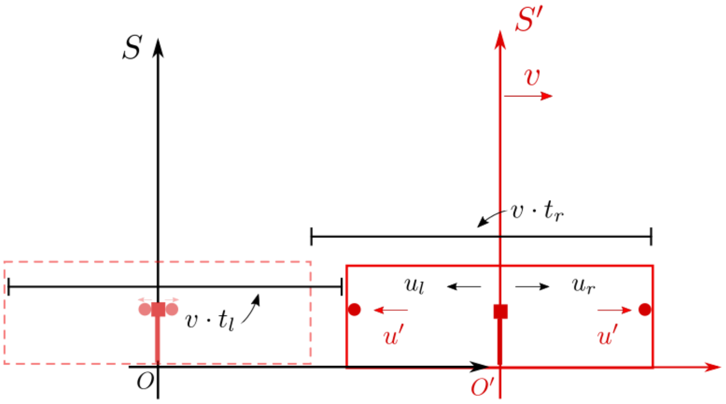

Figure 2: Time measurements

at some undefined speed. In fact the calculation is much easier if you do this, because the velocity of light c is the same in both directions!

(a) Show that now $t_r = (L'/(2\,\gamma) + v\,t_r)/c$ and $t_l = (L'/(2\,\gamma) - v\,t_l)/c$.

(b) Thus show that $\Delta t = t_r - t_l = \gamma\, L'\, v/c^2$, which is of course exactly what you got earlier.

The nature of the object/signal has nothing to do with the time interval here; it just helps us to imagine the process. You will see that more clearly in the next question.

4. Let us practice using the Δ notation. Thus we can write $L' = \Delta x'$, $\Delta t = t_r - t_l$, and so on.

   (a) Convince yourself that simultaneity in S' means $\Delta t' = 0$.

   (b) Now use the Lorentz transformations to show that in the physical situation considered in the last question we have $\Delta t = \gamma\, \Delta x'\, v/c^2$. Now substitute for $\Delta x'$ to get the result obtained in the last question.

   Notice that when you write Δt as $\gamma\, \Delta x'\, v/c^2$, you see clearly why the time interval is independent of the object/signal that is used to probe it.

5. Now go back to the first question. The coordinates of the end of the rod are measured simultaneously in S.

   (a) Show that the two measurements are not simultaneous in S'; find the duration between the events.

   (b) The length of the rod in S' is nevertheless equal to $\Delta x'$. Why?

6. Let us modify our thought experiment in the question above so that the light ray starts out at one end, reflects off the wall, and returns to the point of emission, as shown in Figure 4.

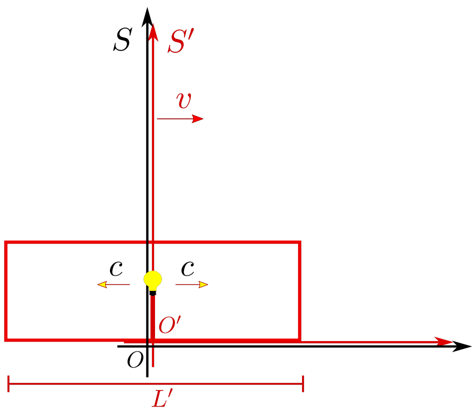

Figure 3: Light signals and simultaneity

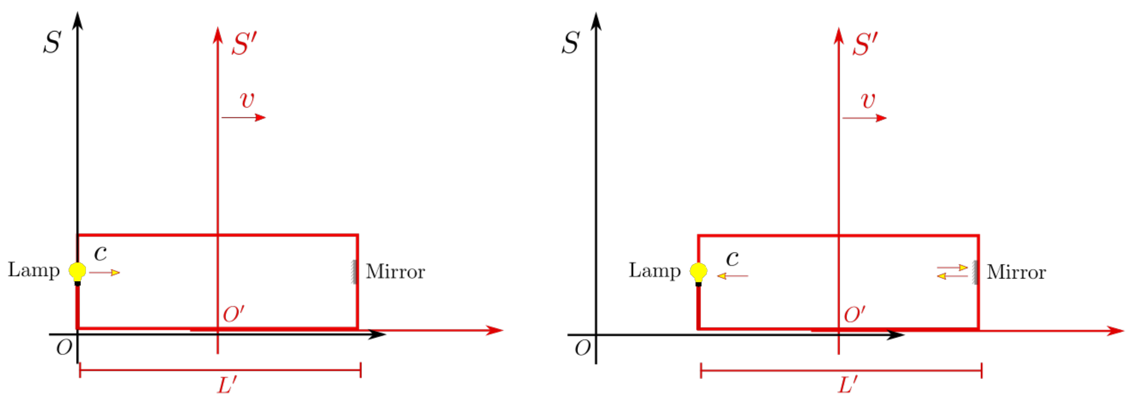

Figure 4: Light signals and proper time

(a) Show that in the rest frame of the box, the time taken for the round trip is $\Delta t' = 2\,L'/c$.This is called a *proper* time interval – it is special because there is no change in the position coordinate between the first event (emission) and the second (detection). It is the time interval that would be measured by a stationary clock.

(b) Show that the Δt corresponding to this pair of events is $\gamma\,\Delta t'$. Since $\gamma > 1$ we call this *time dilation*.

(c) Convince yourself that the order of the events remains the same in all frames, so long as the velocity of the rest frame is less than c. If this were not true, we would have bizarre situations like events occurring in one sequence in one frame and in the opposite sequence in another frame, making non-sense of cause and effect. No objects/signals/influences ever travel at speeds greater than that of light.

7. In the common-sense world, the length of an object L remains the same in all frames of

reference. We would like to discover if there is anything that remains constant in all frames of reference. Show that $\Delta s^2 = -c^2 \Delta t^2 + L^2$ is the same in all frames of reference. Notice that for light Δs, which is called "interval" within the framework of Einstein, is 0 in all frames of reference; this is just another way of saying that the velocity of light is the same in all frames of reference.

8. One very nice way to represent the evolution of an event in time is to draw what is called a world-line on a space-time diagram. This diagram has $c\,t$ on the vertical axis and x on the horizontal axis. The diagram (see Figure 5) below shows the world-line for an object at rest in the frame shown, at a point x_1. Stare at the diagram for a while and you'll get it. Draw the world-lines for the following situations:
   (a) an object at rest at some other point x_2;
   (b) an object moving at velocity v.
   (c) a light ray moving to the right and another light ray moving to the left.

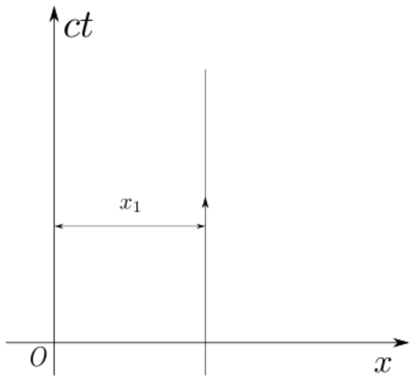

Figure 5: World-line for an object at rest

Notice that the two light rays mark out a cone, called the *light cone*. If any event occurs at the origin of the light cone, it can influence events only within the light cone, since otherwise the influence would have to travel faster than light.

9. We will now study something slightly more difficult: the composition (combination) of Lorentz transformations. Consider three inertial frames of reference, S_o, S_1, and S_2. S_1 moves with respect to S_o at v_{10}; S_2 moves with respect to S_o at v_{20} and with respect to S_1 at v_{21}. We thus have three γs: $\gamma_{10} = 1/\sqrt{1 - v_{10}^2/c^2}$, $\gamma_{20} = 1/\sqrt{1 - v_{20}^2/c^2}$, and $\gamma_{21} = 1/\sqrt{1 - v_{21}^2/c^2}$.

   Show that
$$\gamma_{20} = \gamma_{21}\,\gamma_{10}\left(1 + \frac{v_{21}\,v_{10}}{c^2}\right) \tag{4}$$

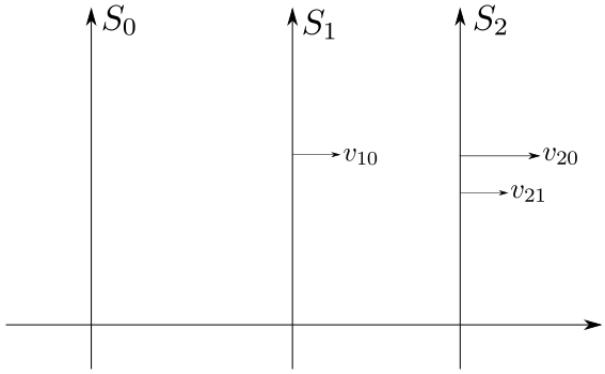

Figure 6: Composition of Lorentz Transformations

If you know how to handle matrices and use the matrix representation of the Lorentz transformations given in the last problem of the notes, you can do this very easily; otherwise it will involve a lot of algebra.

10. Consider a rod at rest in frame S_o. Its length in this frame is L'. Determine its length as observed in S_1 and S_2.What is L_2/L_1? Notice that it is not equal to γ_{21}. In other words the length contraction factor of an object in any frame is $1/\gamma$ only when it is at rest in one of the frames. This subtle point has very important ramifications.

11. Imagine an infinite wire carrying electric charge distributed uniformly, with charge per unit length λ_o in S_o, λ_1 in S_1, and λ_2 in S_2. Assuming that electric charge appears the same in all frames (which is known to be true), how are the three λs related? The answer follows directly from the last question. Think of the infinite wire as a series of rods, like the one in the last question. Since each rod undergoes length contraction, the answer follows.

12. Imagine two infinite charged wires, one with positive charge and the other with negative charge. In a certain frame the positive wire is at rest, and the negative wire is moving to the right with velocity u. In this frame the density of positive charge is exactly equal and opposite to the density of negative charges, so that the net charge density is 0. Now look at the system from a frame moving along the wire with a velocity v. Do the positive and negative charges still cancel each other out?