Lecture 8 – The Quantum World

Physics for Pedestrians

8th August, 2019

1 The Problems with Classical Physics

1.1 The Atom

As explained in the last class, there seemed to be a pleasing symmetry between the physics of the large scale (the solar system) and that of the small scale (the atom). Both of these systems had very similar (albeit different) forces acting on them.

The force that held the solar system was gravity, and Newton told us that

$$\begin{aligned} F_g &= \frac{GM_sM}{r^2} = \frac{D}{r^2} \\ \text{where} D &= GM_sM \end{aligned} \tag{1}$$

From the studies on Electromagnetism in the 1800s (which I have sadly skipped throughout this course), it was found that there existed another much stronger force that affected charged particles, known as the *Couloumb Force*:

$$\begin{aligned} F_e &= \frac{K_eq_1q_2}{r^2} = \frac{D}{r^2} \\ \text{where} D &= K_eq_1q_2 \end{aligned} \tag{2}$$

Now, since the forces looked similar, so should the solutions. From Newton's analyses, we know that the planets take orbits around the sun that are elliptical, but very nearly circular. Their distance from the sun can thus be given by the following equation

$$L = \frac{L^2}{DM} \tag{3}$$

For the enthusiastic reader

The actual orbits are elliptical, and their solutions are actually given by

$$r = \frac{L^2/DM}{1+\epsilon\cos\theta}$$

The term ϵ can then be taken to 0 if the orbit is circular.

The quantity L is angular momentum, a conserved quantity we discussed earlier that depends on an objects momentum, and its distance from an axis of rotation. It is the quantity that is conserved for an object that is rotating.

$$L = mvr \tag{4}$$

Since atoms are known to be spherical, the same equation holds there as well. Thus, if one knows the angular momentum of the atom, one could just as well solve for the radius of an atom

$$\begin{aligned} r &= \frac{L^2}{DM} \\ D &= K_e e^2 \quad e \text{ is the charge of the electron} \end{aligned} \tag{5}$$

We just need to cover one last thing now: energy. In a circular orbit, the kinetic energy of the planet (or electron) is half the potential energy. And the potential energy is given by

$$\text{P.E.} = -\frac{D}{r}$$

Thus, the total energy (using the fact that K.E. = 1/2 P.E.

$$E = \text{K.E.} + \text{P.E.} = -\frac{1}{2}\frac{D}{r} \tag{6}$$

This, again, should be true for **both** the planets around the solar system as well as atoms.

However, there is a clear difference:

Accelerated charges emit electromagnetic radiation (light).

And so, as the electron orbits around the nucleus, it is clearly accelerating, meaning that it radiates energy. However, in doing so, its total energy reduces, which means that its radius **should also reduce**! The classical atom is not stable: electrons will eventually fall into the nucleus!

Question: Show that as $E\uparrow$, $r\downarrow$.

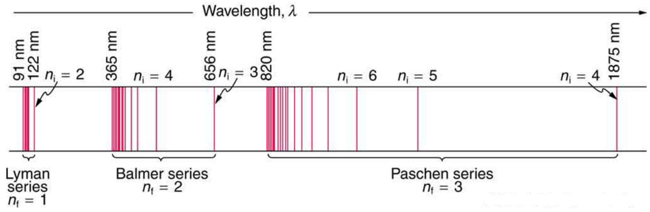

Figure 1: The emission lines of Hydrogen: they are discrete, meaning that between any two lines there is a gap with no light emitted. This is very different from the spectrum that you see when you split the light of the sun, for example.

So, in the classical atom, the electron would then be expected to radiate a continuous range of colours ("frequencies") as it fell into the proton. It seems as if classical physics predicts that we cannot exist![1]

1.2 Discrete Spectra

As if that wasn't enough, when even the simplest atoms were observed, it was found that they **didn't** have continuous spectra. Instead, they seemed to emit light in specific frequencies, shown in Figure (1). Johann Balmer, a Swiss mathematician and schoolteacher, came up with an empirical formula that described the positions of these lines qualitatively. Johannes Rydberg, a Swedish physicist generalised this formula to

$$\frac{1}{\lambda} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \tag{7}$$

where λ is the wavelength of light emitted, and n and m are *any* two integers, and R is a constant called the Rydberg constant, which was known experimentally to great accuracy. Every wavelength seemed to be associated to a pair of integers, and there seemed to be no reason for this.

$$R = 1.097 \times 10^7 \mathrm{m}^{-1}$$

Using this formula, people could *predict* where lines would be found, and sure enough they were right. Was this a new law? Would R be found to be a new constant of Nature, just as h was? It worked, but **why**?

2 Bohr's Atom

German physicist Niels Bohr had two radical ideas that he thought would solve this problem:

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[1] But, I hope I will not have to work to convince you, we *do* exist.

1. Maybe there were some **special** orbits in which atoms did not radiate. (Of course, this went completely against Newton's Laws. According to the solar system model of the atom, no orbit should be "special".)
2. Maybe the atoms absorbed and emitted light in discrete amounts, which made the electrons of the atom "jump" between these orbits. (Of course, *this* went against the established theories of Electromagnetism.)

Bohr really wasn't making any friends. So why then were his ideas entertained? It turns out, these assumptions worked rather well to describe the physical world, as we shall see.

2.1 De Broglie's Hypothesis

Louis De Broglie proposed a hypothesis: if light (which acts so much like a wave) could behave as particles do, perhaps other objects that we thought of as "particles" could behave as waves do?

De Broglie attempted to associate a **wavelength** to electrons, which depended on their momentum. This "de Broglie" wavelength is given by

$$\lambda = \frac{h}{mv} \tag{8}$$

This hypothesis, he felt, would answer the question as to why only special orbits were allowed in the atom. Consider waves set up on a string that's fixed at both ends, only some types of waves would be allowed – all those that had a fixed number of wavelengths in between the end points (see Figure (2)). If this were not the case, the waves would interfere destructively and die out. Only waves that interfered "constructively" would survive.

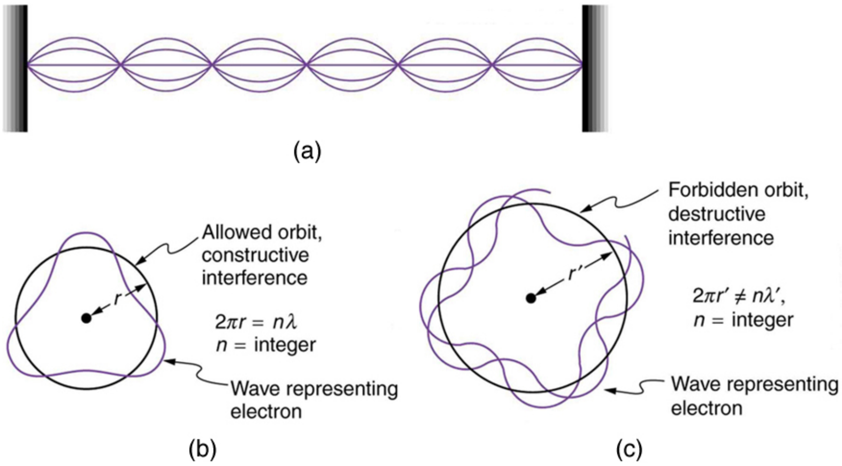

Figure 2: Just as the waves on a string (a) must be an integer number of wavelengths, so de Broglie hypothesised, should the waves in a circular orbit of an atom. (b) If the waves interfered constructively, the orbit would be allowed. (c) If they interfered destructively, the orbit would be forbidden.

Thus, in order for this to work, the circumference of these special orbits should be related to their de Broglie wavelengths by (at most) an integer. Thus,

$$\begin{aligned} 2\pi r_n &= n\lambda \\ &= n\frac{h}{mv} \\ \implies mvr_n &= n\left(\frac{h}{2\pi}\right) \\ L_n &= n\hbar \end{aligned} \tag{9}$$

The quantity $h/(2\pi)$ appears so often in this subject that it has it's own name, $\hbar$.[2] From the above calculation (which you should be able to reproduce), it looks as if the angular momentum is **quantised**, meaning that it comes in little lumps of $\hbar$, $2\hbar$, $3\hbar$, and so on. There is no possibility of getting angular momentum of $1.3\hbar$.

With this, we can now figure out the radius of a circular orbit (assuming our solar-system model) from Equation (3).

$$\begin{aligned} r_n &= \frac{L_n^2}{DM} \\ &= \frac{n^2\hbar^2}{DM} \end{aligned} \tag{10}$$

Question: Show that the first orbit has a radius

$$r_1 = \frac{\hbar^2}{DM} \tag{11}$$

Question: Show that any other orbit r_n is given by

$$r_n = n^2 r_1 \tag{12}$$

The natural unit of length for the atomic scale is thus r_1 which is called the **Bohr radius**. Plugging in the values of the different constants, we can find that

$$r_1 \approx 0.5 \times 10^{10} m = 0.5 \text{ Å}$$

The unit Angstrom, denoted by Å, is 10^{-10} m, and is the natural unit of length when dealing with phenomena at the atomic scale.

Thus, de Broglie's hypothesis had explained the first of Bohr's assumptions: electrons seem to live in certain privileged orbits, since these are the orbits where their "waves" interfere constructively. What was left was to explain the discrete spectrum.

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[2] Pronounced "h-bar".

2.2 Leap-frogging electrons

Let us now consider the energy of these orbits. From Equation (6), we know that the energy depends on the radius, and thus for every orbit with radius $r_n = n^2 r_1$, there is a specific energy.

$$E_n = -\frac{D}{r_n} \tag{13}$$

Question: Show that

$$E_n = -\frac{D^2 M}{n^2 \hbar^2} \tag{14}$$

We are now ready to explain Rydberg's formula. Suppose, as Einstein had already shown, that light is absorbed or emitted in chunks of energy $E = hf$. Atoms would only be able to emit (or absorb) certain frequencies when their electrons leap from one orbit to the other. An electron from a higher (or more "excited") orbit would fall to a lower orbit and emit radiation, and could absorb a photon and jump to a higher energetic orbit, or could emit a photon and jump to an orbit with lower energy.

Question: Show that if an electron jumps from a higher orbit with energy E_n to a lower orbit with energy E_m, and if all this energy is converted into light, then the light is emitted with a frequency given by

$$f = \frac{\Delta E}{h} = \frac{E_n - E_m}{h} \tag{15}$$

Question: What would happen if it was jumping from a lower to a higher orbit?
Answer: The earlier number would be negative, but that would just mean that it was **absorbing** a photon, instead of emitting it.

Question: Plug in the value of E_n and E_m from Equation (14) to find that:

$$f = \frac{D^2 M}{h \times \hbar^2} \left(\frac{1}{m^2} - \frac{1}{n^2} \right) \tag{16}$$

Question: Relating the frequency of light f to its wavelength λ and speed c, show that

$$\frac{1}{\lambda} = \underbrace{\frac{D^2 M}{hc\hbar^2}}_{\text{Rydberg constant } (R)} \left(\frac{1}{m^2} - \frac{1}{n^2} \right) \tag{17}$$

For the enthusiastic reader

Question: By plugging in the values of the different physical constants, show that the numerical values of R agree.

Thus, we have shown *why* the empirical relation given by Rydberg works, and we have explained his "constant" in terms of more fundamental constants of Nature.

It is in this sense that scientists try to unify theories: if the constant parameters (which you have no liberty to change) in one theory can be described in terms of the constant parameters of some other theory, then we say that the first theory is a *result* of the second: the two have been *unified*.

3 The Spooky World of Quantum Mechanics

Now you may not truly believe that electrons behave like waves. Let's ask ourselves what this would mean. Very much as with light, a deciding factor between whether it behaves like a classical particle or classical wave would be some sort of double-slit experiment. This is exactly what Feynman proposed, and was actually carried out by Akira Tonomura and his team at Hitachi in 1989. (You can see the video here.) The result is astouning: electrons behave like waves of a wavelength exactly predicted by de Broglie:

$$\lambda = \frac{h}{p}$$

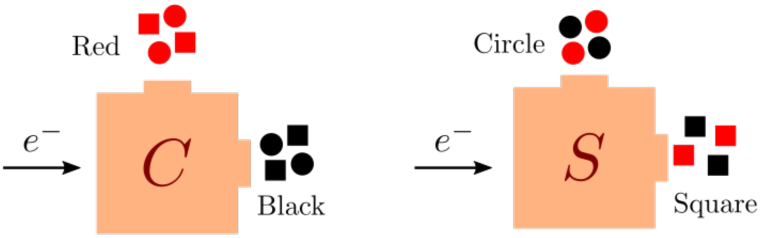
This is only strange if you consider the electrons as going through one or the other slit. **This idea is inconsistent with experiment.** It turns out that there is only one consistent way to describe this system and what is happening in it, and it's not easy.

We'll try to describe the weirder aspects of it, and see what exactly this means. I must warn you, though: it's an unsettling story, perhaps the most unsettling story that's ever come out of physical experiments.

Consider a bunch of happy little electrons, going about their own business. Let us say that these electrons have two properties, shape and colour. They can be either square (🟥, ■) or circular (●, 🔴), or red (⧇) or black (⧇).

The experiments I'm going to talk about have all actually been carried out, though not necessarily with electrons. The big daddy, so to speak, of these experiments is called the Stern-Gerlach experiment. Look it up if you're interested.

Let us now say that someone has made us "colour" and "shape" machines that can sort these electrons for us (see Figure (3).)



(a) A colour sorting box (b) A shape sorting box

Figure 3: Colour and shape sorting boxes: electrons that come out of these boxes are found to all be (a) either red or black, or (b) either circle or square. In every case, the ratio is 50:50.

Experiments are performed to find out how many electrons are black and how many red, and it is found that half of all the electrons sent into the box are black, and half red. The same result is found with the shape box, where half are found to be circular and half square.

It is important to mention at this juncture that the actual mechanisms of the box **do not matter at all**: scientists have made many such boxes in very very different ways, and the results have always been the same.

A natural question to ask at this juncture is the following: are colour and shape correlated? This can be resolved in an easy way: consider all possible colour and shape combinations: pass an electron first through a colour box, and then through a shape box, and count the different ratios of both types. Then do the same thing by passing it through a shape box and a colour box. The results, in each case, show that there are always 50% of each type. It *seems* that there is no correlation between colour and shape.

Suppose now we take the electrons that leave the black end of a colour box, and pass them through another colour box, as shown in Figure (4), then they are *all* found to be black. Indeed, this is the **meaning** of a colour box. The same thing happens with a shape box.

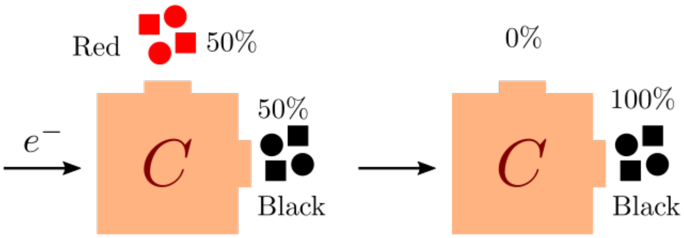


Figure 4: Of all the electrons produced, 50% are red and 50% black. If the 50% are found to be black are passed through another colour box, they are *all* found to be black.

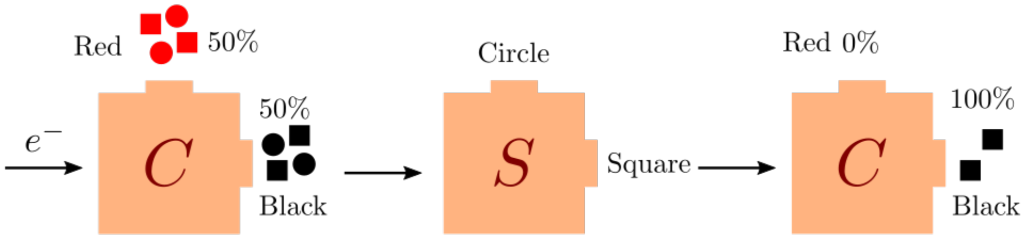
The percentages given in the figures all represent the percentage with respect to the number of electrons that entered **that detector**, *not* the total number of electrons.

Everyone ok so far? Good. Now let's consider three different experiments that go from simpler to more complicated. In each of the cases below, the electrons are sent in **one at a time**, they don't interact with each other.

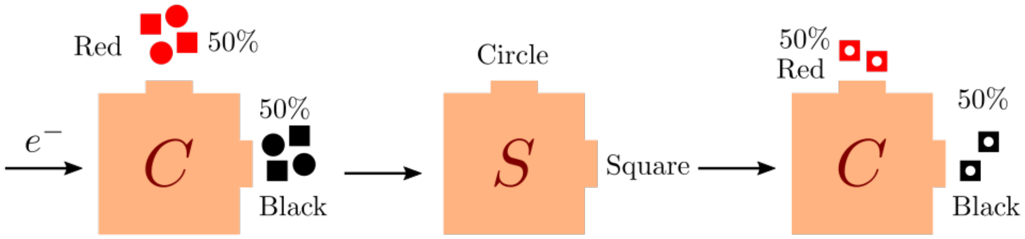
3.1 Experiment 1

Consider a case where we pass the electrons one by one through a colour box, and then collect only the black electrons and pass them through a shape box. We then pass only those of the resulting electrons (which should all be square) through *another* colour box. What would you expect?

You would probably reason that since the electrons you sent through the shape box were black, then those that exited the shape box were also black (albeit square), and so when you measure them again you'd expect them to all be black. However, *you'd be wrong.* What actually happens is that we get 50% each of black and red electrons!



(a) What you would expect to happen in Experiment 1



(b) What actually happens in Experiment 1

Figure 5: (a) The results one would expect from Experiment 1, and (b) what actually happens. **Important:** percentages given next to boxes are with respect to the number of electrons that enter *that box*. The conclusion is simple: shape measurements seem to randomise colour.

This is very strange, as all we have done in this case (which differs from the last) is place a shape box in between two colour boxes. And the result has changed! It seems that shape and colour *are* correlated!

Remember, this works even when the electrons are sent in one by one! The measurement of shape *randomises* the colour of each electron.

3.2 Experiment 2

Let's now move to a slightly more complicated problem: electrons are passed first through a colour box, and the black ones are passed then through a shape box. Both shapes move along different paths, "bounce" off mirrors and are recombined in a "beam-splitter"[3] and passed through another shape box.

What do you think would happen? Well, since we've seen that when electrons of a certain shape enter a colour box, their colours are 50/50, perhaps this means that when electrons of a certain *colour* enter a shape box, their *shapes* are 50/50. In other words, perhaps there are 50% of squares and 50% of circles, which then recombine at the beam-splitter, giving again 50/50 of square and circle. And indeed, this is exactly what we see.

Whew! Our intuition is not *completely* wrong then!

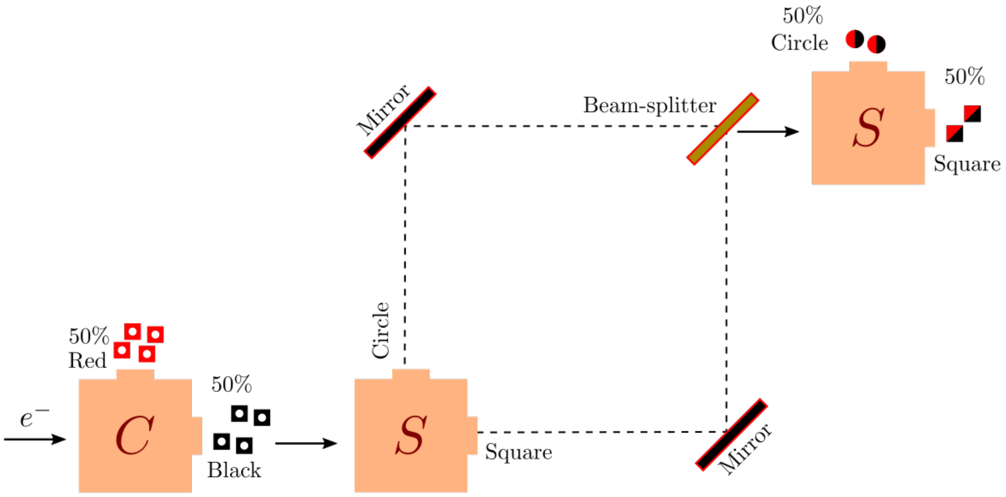


Figure 6: What you would expect to happen (and what does actually happen) in Experiment 2: the colour measurement randomises the shapes.

Feeling a little better? Well here's where it gets *really* weird.

3.3 Experiment 3a

Let's now change the measurement at the end to a colour measurement. What now? Well, since we seem to think that shape measurements randomise colour (as we've seen – or rather, think we've seen – in the previous experiment), it wouldn't be too far a stretch to imagine that the result would give us 50/50 of red and black.

Too bad! We get 100% of black electrons! Not a single red electron is found. Black electrons went into the first shape box, and only black electrons were found after the two different shapes were recombined!

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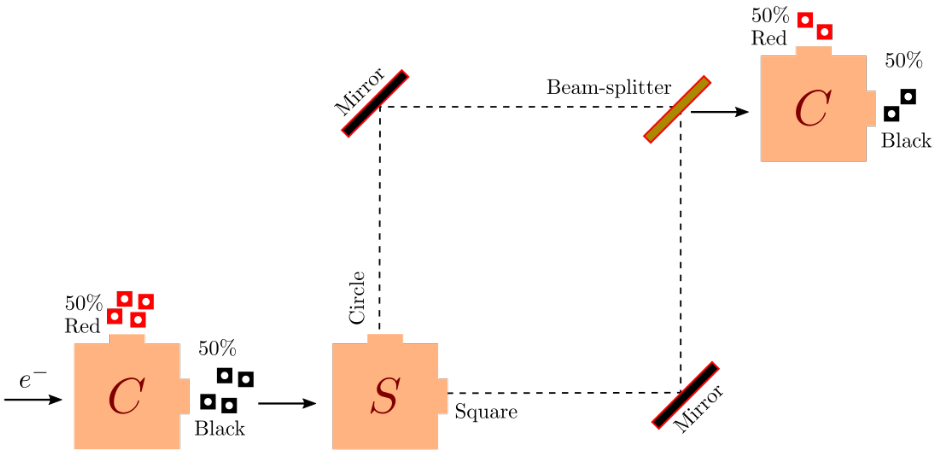
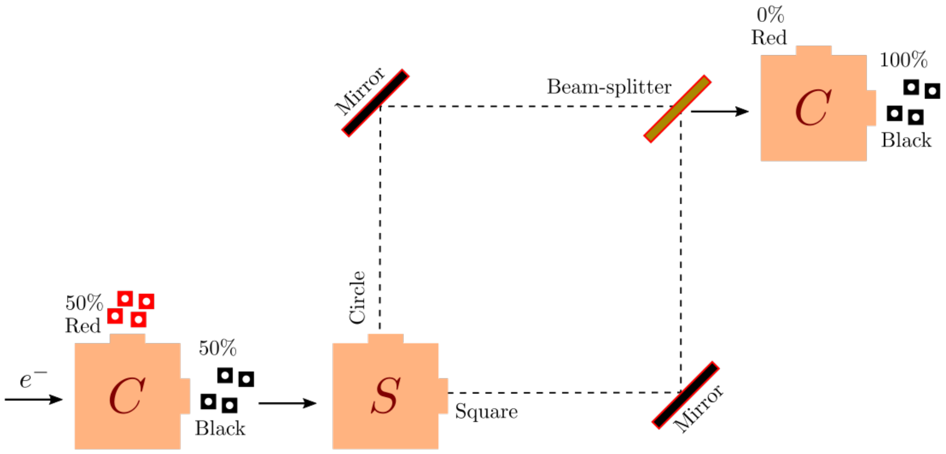
[3] Don't worry about what this is, it's just a complicated mirror that joins two beams and sends out one. The mechanisms of these mirrors and beam-splitters are unimportant: the experiment works the same way no matter what.



(a) What you might expect to happen in Experiment 3a



(b) What actually happens in Experiment 3a

Figure 7: Despite measuring the shape, when the colour is measured again (after the different shapes are recombined and hence "unknown" once more) all the electrons are black, as you would naively have supposed!

So you see that what we've seen so far is even more puzzling than we thought. It's not just that the measurement of shape randomises colour and the measurement of colour randomises shape, it's far more insidious: knowledge of the shape precludes knowledge of the colour, and knowledge of the colour precludes knowledge of the shape.

Since in Experiment 3(a) we couldn't know the shape of the electrons (since they were both mixed at the beam splitter) we were able to know everything about their colour!

Question: Go back and look at the other experiments using this idea. Make sure you understand how all their results can be explained using this principle: *knowing*[a] the shapes means that the colour is random, and knowing the colour means that the shapes are random.

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[a] As opposed to simply measuring.

3.4 Experiment 3b

You're annoyed by this, and rightly so. You decide to push Nature into a corner, because she's really misbehaving rather badly. You take the same experiment that you just carried out (Experiment 3a) and now you place a wall along one of the arms. Furthermore, you don't want these sneaky electrons talking to each other, and so you make the arms millions of kilometres long, so that no information about the wall can get to the electron going along the other arm (without the wall).

Let's think this through: in this case, we *know* that the electrons that reach the beam-splitter are all circular (since the square path has been blocked). From our earlier analysis, you should realise that if we *know* the electron's shape, then its colour should be completely random. And this is indeed the case! We could do the same experiment, with the wall along the circular path and (happily) we'd get the same answer.

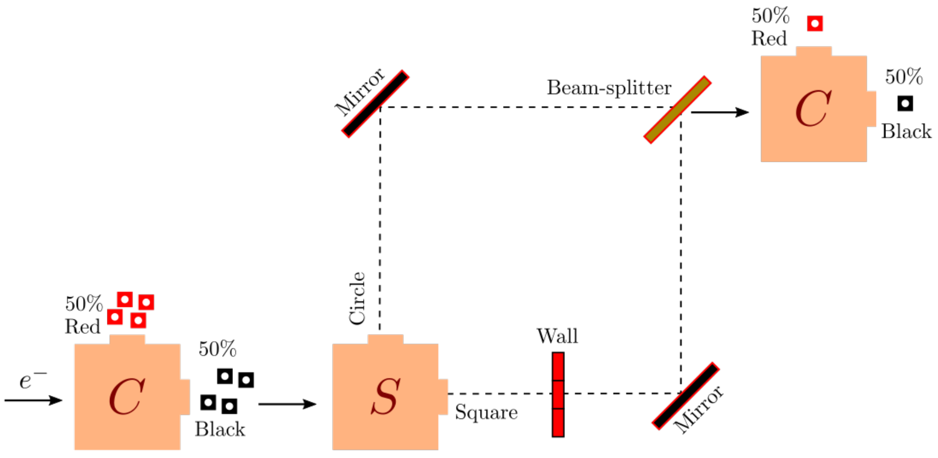
But remember, we're sending **one electron in at a time!** So what did this electron do? Well, let's examine the possibilities that logic allows us:

1. **It took the circular path:** Well, this is not possible, since if all electrons did this, there would be no difference between the experiments with and without the wall (but there certainly is!).
2. **It took the square path:** Well, that's now possible either, since the square path was walled up.
3. **It took both paths:** This is tempting: perhaps our sneaky little electrons "split" into two, and took both paths simultaneous. The problem is, we could put a detector along the paths and we would **never** find half-an-electron. Electrons come in fixed lumps, and so it must have taken one path or the other.
4. **It took neither path:** This is just silly, I hope you don't need me to explain why this isn't the case, but even so: if we walled up both the arms, no electrons would be detected at all.

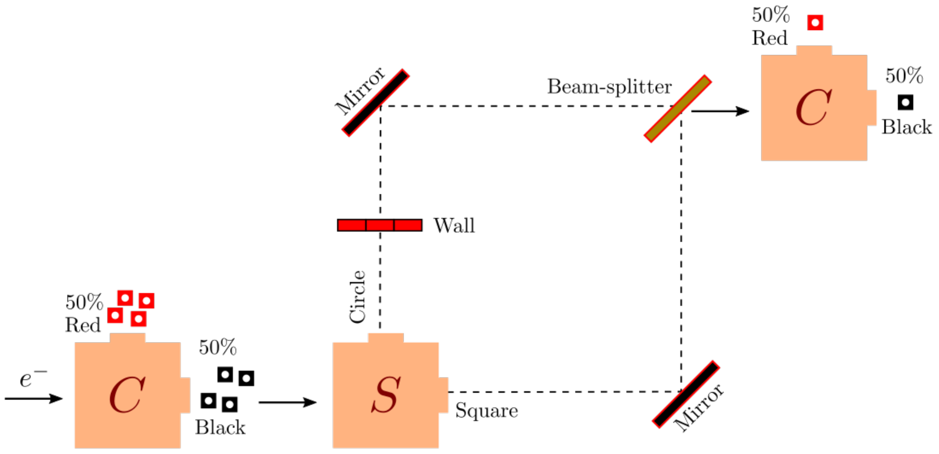
So what is the right answer? It looks like all logical possibilities have been exhausted. And indeed – in our current way of looking at the world – they have. The only answer we can give is that in between measurements, the electron was doing something strange: it was in a *superposition* of going along the square path and the circular path *at the same time.*

3.5 Superpositions

So far, the word *superposition* is just a fancy way of saying "We don't really know what it's doing". Let's use it in a sentence: in Experiment 1, the measurement of the electrons state caused it to be in a superposition of red and black states.



(a) Placing a wall in the square path randomises the results of colour.



(b) Placing a wall in the circle path *also* randomises the colour.

Figure 8: Experiment 3a with a slight modification: a wall is placed in one of the arms, and the arms are made so long that the electrons can't communicate with each other. Analysing this 'classically', we exhaust all our logical possibilities. We thus say that the electron is in a *superposition* of both paths.

Let's try to put some mathematics to this: it turns out in order to do this, we need to speak of something called the **state** of a system. The electron may be in a state of colour (red or black) or in a state of shape (square or circle) but the knowledge of the electrons colour places the electron into a superposition of shape states, and the knowledge of an electron's shape forces it into a superposition of colour states.

Let's denote the state by this little symbol $|\text{state}\rangle$.[4] Since the superposition looks a little like a

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[4] Pronounced "ket". This notation is due to Dirac, and is very useful to physicists. It is also an attempt at physicist

"sum" of two states, let's be bold and denote it by the following: say that the state "red" is a superposition of square and circle:

$$\left|\textcolor{red}{\boxdot}\right\rangle = |\blacksquare\rangle + |\bullet\rangle$$

Of course, the state "black" is *also* a superposition of square and circle, but it cannot be the *same* superposition as then it would be the same state. We guess that perhaps it could be a − sign in between the terms:

$$\left|\boxdot\right\rangle = |\blacksquare\rangle - |\bullet\rangle$$

You can possibly now understand why we used the "+" and "−" signs. We could now try to *add* the two equations given above to find how the $|\blacksquare\rangle$ state can be written as a superposition of $\left|\textcolor{red}{\boxdot}\right\rangle$ and $\left|\boxdot\right\rangle$!

Question: Show that by adding the two equations given above you get that

$$|\blacksquare\rangle = \frac{1}{2}\left|\textcolor{red}{\boxdot}\right\rangle + \frac{1}{2}\left|\boxdot\right\rangle$$

Of course, the factor of half that comes in front of the equation is distasteful: the states of colour and shape are equivalent (as we have seen with our experiments), and so it would be nice if this symmetry manifested itself in the mathematics.

Question: Show that if – instead of the earlier equation we'd written – we had chosen the following for the colour states:

$$\begin{aligned}\left|\textcolor{red}{\boxdot}\right\rangle &= \frac{1}{\sqrt{2}}|\blacksquare\rangle + \frac{1}{\sqrt{2}}|\bullet\rangle \\ \left|\boxdot\right\rangle &= \frac{1}{\sqrt{2}}|\blacksquare\rangle - \frac{1}{\sqrt{2}}|\bullet\rangle\end{aligned} \tag{18}$$

Then the corresponding equations for the shape states are

$$\begin{aligned}|\blacksquare\rangle &= \frac{1}{\sqrt{2}}\left|\textcolor{red}{\boxdot}\right\rangle + \frac{1}{\sqrt{2}}\left|\boxdot\right\rangle \\ |\bullet\rangle &= \frac{1}{\sqrt{2}}\left|\textcolor{red}{\boxdot}\right\rangle - \frac{1}{\sqrt{2}}\left|\boxdot\right\rangle\end{aligned} \tag{19}$$

(This question might seem a lot harder it actually is. It's just addition, make sure you do it.)

In general, an electron can be in any state which is a combination of the above. This is what we call (mathematically, at least, since other language fails us), a superposition. Thus, any state, for example,

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humour. The "ket" has an equivalent double who appears very often, called the "bra", denoted by ⟨state|. I will leave it to you to figure out what the quantity $\langle \text{state}_1|\text{state}_2\rangle$ (which appears often), is called.

$$|\psi\rangle = a\left|\boxed{\circ}\right\rangle + b\,|\blacksquare\rangle$$

is allowed. These constants a and b are known as the probability amplitudes, as Feynman discusses in the reading. Their name again is a result of our ignorance. We've found that these quantities seem to add in ways we understand.

In the case of waves, the heights (or *amplitudes*) of the waves added, but their *intensities* didn't. With matter waves (like electrons) we found that the probabilities of their detection didn't add. But these constants (a and b) do, and so we call them *probability amplitudes*.

The probabilities of an electron being in a state can be found by squaring these numbers (just as the intensities of classical waves can be got by squaring the amplitudes).

Thus, when an electron is in the "square" state, it has a 50% chance $\left((\frac{1}{\sqrt{2}})^2\right)$ of being red and the same of being black!

Question: Go back and try to explain all the results of the experiments we conducted using these "states" and probabilities.