Assignment 0:
Sample Questions and Answers

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1 Dimensional Analysis

Question: Consider a bob of mass m attached to a massless string of some length l on the surface of the Earth. When released from some angle θ_0, the the bob is found to oscillate. Find a dimensionally correct formula for the time period of oscillation T.

Answer: We are looking for a formula that relates the time period of oscillation T to the parameters in the problem. Which parameters could T depend on? These parameters must be specific to the system, and so it could depend on

- **The length of the pendulum** l, since if the length of the pendulum were zero, I would not expect any oscillation at all, and so I would expect its time period to be zero. (Experimentally, this is quite obvious: as you reduce the length of the pendulum, it swings faster, making the time for an oscillation smaller. **Dimension:** $[l] = \mathbf{M}^0\,\mathbf{L}^1\,\mathbf{T}^0$
- **The acceleration due to gravity** g, since the main reason for the pendulum's oscillation is due to the fact that gravity acts on the bob, pulling it down. There must thus be *some* quantity in the formula that encodes this fact. In free space (in the absence of g) I would not expect it to change its position at all. **Dimension:** $[g] = \mathbf{M}^0\,\mathbf{L}^1\,\mathbf{T}^{-2}$
- **The mass of the bob** m, since we have established that the fundamental force responsible for the bob's motion is gravity, and the gravitational force is dependent on the object's mass. **Dimension:** $[m] = \mathbf{M}^1\,\mathbf{L}^0\,\mathbf{T}^0$
- **The initial angle of release** θ_0, as it is conceivable that the further away we release the bob, the further it will have to travel to get back to where it was, meaning that the time it takes could possibly be larger. **Dimension:**[1] $[\theta_0] = \mathbf{M}^0\,\mathbf{L}^0\,\mathbf{T}^0$

Now, the time period is some combination of these parameters. Since they all have different dimensions, a statement such as "Time Period = $l+g+\theta_0$" is nonsense (or *dimensionally incorrect*), as you cannot add objects of different dimensions. The same goes for adding different powers: "Time Period = $l^3 + m^2 + g^4$" is nonsense too, for the same reason. One possibility then is to find

[1] Angles are ratios of lengths, and so are dimensionless.

some combination of their powers, for example

$$\text{Time Period} = C \times l^a \times g^b \times m^c \times \theta_0^d,$$

where C is a *dimensionless number* that we cannot determine using this method. We can now look at the dimensions of the above equation (by which I mean we ignore all the numerical values, and we focus only on the dimensions on either side. For example, it doesn't matter if the length of the pendulum is 5 cm or 3 m, and so on. What follows should hold *independently* of these numerical values). We have:

$$[\text{Time Period}] = [l]^a \times [g]^b \times [m]^c \times [\theta_0]^d$$

Plugging in the dimensions of each of the quantities on the right hand side, we have

$$T = \left(L^a\right) \times \left(L^b T^{-2b}\right) \times \left(M^c\right) \times \left(1^d\right),$$

The first thing you should notice is that since θ_0 is dimensionless, the value of d would not affect the above equation at all. By enforcing that the dimensions on either side be the same, we have

$$\begin{aligned} a + b &= 0 \\ -2b &= 1 \\ c &= 0 \end{aligned}$$

Solving the above equations, it's clear that $a = 1/2, b = -1/2, c = 0$ is the solution. Thus, we could construct a quantity of dimension time out of these parameters:

$$T = \sqrt{\frac{l}{g}}.$$

Of course, you'd object, since we have completely ignored θ_0. Since it's dimensionless, we could always multiply the above equation with θ_0, and it would remain dimensionally acceptable. But then I could also multiply it with θ_0^2, or any other function of θ_0, and the equation would continue to be dimensionally acceptable. Thus, the general formula for the time period *could* be

$$\text{Time Period } = \sqrt{\frac{l}{g}} \times f(\theta_0),$$

where f is some arbitrary function. Curiously, it turns out that this is indeed the case.

Question: Let us consider the simple harmonic oscillator we saw in class.

(a) Write out the differential equation that it satisfies, and identify a time scale associated with this problem. Call this τ_1.

(b) Write a code to solve this system using the leapfrog method, setting $\tau_1 = 1$, with initial conditions $x(0) = 1,\ v(0) = 0$.

(c) Plot graphs of the solution for different values of $\tau_1 = 1$, $\tau_1 = 2$, and $\tau_1 = 3$ and comment on the graphs obtained.

Answer:

(a) From Newton's Second Law, we know that

$$a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{F}{m}.$$

The spring obeys Hooke's law, and so $F = -kx$. Thus, the differential equation is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{k}{m}x.$$

The time scale associated with the problem can depend on the constants k and m. We know that the dimensions of k are MT^{-2}, and the dimensions of m are M. Thus, we would like to find a formula that is dimensionally correct of the form:

$$\tau_1 = m^a k^b.$$

It should now be simple to show that if $a = 1/2$ and $b = -1/2$, this is satisfied.

Thus, the quantity $\sqrt{m/k}$ has dimensions of time, and we will call this the *time-scale* of the problem, τ_1. In terms of this, the differential equation becomes:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{1}{\tau_1^2}x. \tag{1}$$

(b) The code that is required for this part of the question is almost identical to what was done in the Discussion Session, and so I will not explain it in detail. The only difference is that the acceleration function that we defined is slightly different. In the code we discussed in class, the acceleration function was defined as follows:

```
def a(x,v):
    return -x
```

This corresponds to the case when $\tau_1 = 1$, meaning that we have chosen a unit of time such that τ_1 has the numerical value of 1. Thus, one is solving the differential equation [2]

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -x.$$

What we need to do is simply include τ_1. From Equation (1), we know that the acceleration is given by $-x/\tau_1^2$, so we define our function slightly to include it:

```
def a(x,v):
    return -x/(t1**2)
```

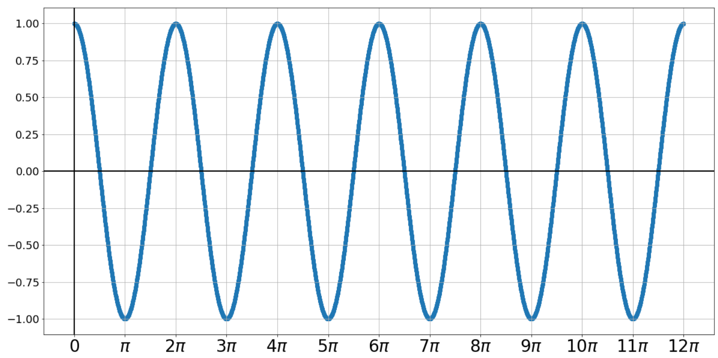
where the variable `t1` is τ_1, and needs to be defined somewhere else in your code, preferably next to your initial conditions. The rest of the code is identical.

(c) The graphs for different values of τ_1 can be seen in Figure (1). By changing τ_1, we are changing the time-period of the oscillator, and thus (as we would expect) the oscillator takes longer to get back to its starting point. However, the general form of the solution remains the same, and in every case, the oscillator completes one oscillation after a time $2\pi \times \tau_1$. You should convince yourselves that all the graphs would look identical if, instead of time t on the x−axis, we used a variable $\eta = t/\tau_1$.

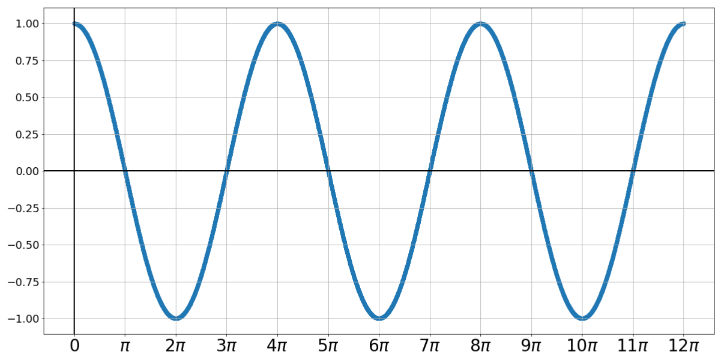
[2] Or equivalently, one is solving the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{1}{\tau_1^2} x, \quad \Longrightarrow \quad \frac{\mathrm{d}^2 x}{\mathrm{d}\eta^2} = -x,$$

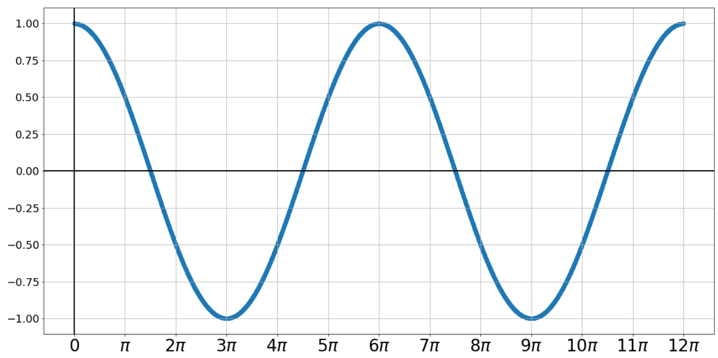
where $\eta = t/\tau_1$, and is a number which measures time in units of τ_1.



(a) $\tau_1 = 1$: The system completes one cycle every $1 \times 2\pi$ seconds.



(b) $\tau_1 = 2$: The system completes one cycle every $2 \times 2\pi$ seconds.



(c) $\tau_1 = 3$: The system completes one cycle every $3 \times 2\pi$ seconds.

Figure 1: The position of the oscillator as function of t, for different values of τ_1.