Assignment 3:
Affine Spaces in Physics

Due: February 20, 2020 (Thursday) **Marks: 15**

1 Affine Transformations

(a) One very nice way to represent the evolution of an event in time is to draw what is called a *world-line* on a space-time diagram. This diagram has t on the vertical axis and x on the horizontal axis. For example, Figure (1) below shows the world-line for an object at rest in some coordinate system, at some point x_1. (Stare at it for a while and you'll get it.) Draw the world-lines for the following situations: **[3]**

(i) An object at rest at some other point x_2.

(ii) An object moving with a velocity v, which is at x_1 when $t = 0$.

(iii) An object moving with a velocity $-v$, which is at x_1 when $t = 0$.

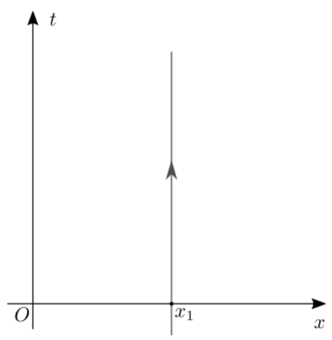

Figure 1: A spacetime diagram for an object at rest: the object's remains at x_1 as time passes by.

(b) Consider the following transformations, and discuss whether they are affine transformations or not. **[2]**

(i) Choosing a different unit of distance to measure the x-axis.

(ii) Choosing a different unit of time to measure the y-axis.

(c) Now suppose the object is (say) a squirrel. You and the squirrel are at a train station: the squirrel is on the platform and you're on a train that is at rest, separated by some distance d metres at some time $t = 0$ seconds. **[5]**

(i) Sketch the squirrel's world-line according to itself and the squirrel's world-line according to you on the same space-time diagram.

(ii) Now imagine the train on which you are begins moving at a constant velocity v. Sketch the same diagram as before.

(iii) In the above two cases, is the transformation that takes you from one to the other *affine*? In other words, is the transformation that takes you from the squirrel's "point-of-view" to your "point-of-view" an affine transformation?

(iv) Now suppose that the train began to accelerate at a constant rate. Sketch the same diagram again. Is the transformation still affine?

(v) Consider the set of world-lines on the platform shown in Figure (2). How do they transform the train starts to move a constant v?

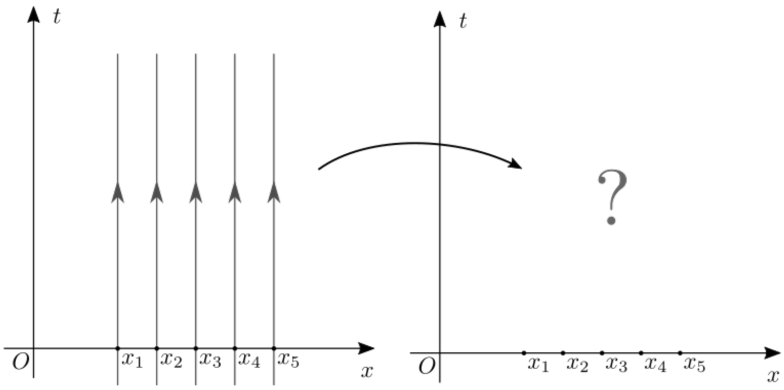

Figure 2: A spacetime diagram for an array of objects on the platform. What is seen from the moving train?

2 Bases in $\mathbb{R}^2$

We saw in class that any vector in $\mathbb{R}^2$ can be described using the (natural) basis $\hat{e}_1$ and $\hat{e}_2$ (shown below). However, we have also seen that this basis is not unique, and that we could choose another basis, for example $\hat{e}'_1$ and $\hat{e}'_2$: **[5]**

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad\qquad \hat{e}'_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \hat{e}'_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(a) Show that $\hat{e}'_1$ and $\hat{e}'_2$ are linearly independent.

(b) Take an arbitrary vector **v** and express it in terms of the $\hat{e}_i$s and $\hat{e}'_i$s respectively:

$$\begin{aligned}\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} &= Ae_1 + Be_2 \\ &= A'e'_1 + B'e'_2.\end{aligned}$$

Determine A, B and A', B' in terms of a and b.

(c) Express $\hat{e}'_1$ in terms of $\hat{e}_1$ and $\hat{e}_2$, and do the same for $\hat{e}'_2$.

(d) Define a *new* basis of your choosing, and repeat the above procedure.