# Assignment 4: Linear Transformations and Matrices

**Due:** February 27, 2020 (Thursday) **Marks: 15**

## 1 Linear Transformations

(a) Consider the following transformations, and check whether they are linear or not. Justify your responses using the defining properties of linear transformations. In the case of a non-linear transformation, give an example of one case where a defining property fails. **[4]**

(i) A transformation $\mathscr{F} : \mathbb{R}^3 \to \mathbb{R}^2$, defined by

$$\mathscr{F}\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 2x_1 + x_3 \\ -4x_2 \end{pmatrix}$$

(ii) A transformation $\mathscr{G} : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$\mathscr{G}\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 4x_1 + 2x_2 \\ 0 \\ x_1 + 3x_3 - 2 \end{pmatrix}$$

(b) Consider the *Rotation matrix* $R(\theta)$ defined as we saw in class, which rotates a vector by an angle $\theta$ counter-clockwise. Show the following properties and interpret the results: **[1]**

(i) $R(\theta_1) \cdot R(\theta_2) = R(\theta_1 + \theta_2)$

(ii) $R(\theta) \cdot R(-\theta) = \mathbb{1}$

## 2 The Metric and the Determinant

(a) In class, we defined the metric to be the usual "Euclidean" one. However, there is no reason for this to be the only metric on a space that satisfies the required properties. Check if the following maps from $\mathbb{R}^2 \to \mathbb{R}$ could constitute a "metric".[1] **[2]**

(i) $g(v, w) = \sqrt{-(v_1 - w_1)^2 + (v_2 - w_2)^2}$

(ii) $g(v, w) = \sqrt{(v_1 - w_1)^3 + (v_2 - w_2)^3}$

[1] You only need to check if the three properties spoken about in the lecture hold.

(iii) $g(v, w) = \sqrt{(v_1 - w_1)^4 + (v_2 - w_2)^4}$

(b) Consider the following transformation $S_1$ that transforms the unit vectors $\hat{e}_1$ and $\hat{e}_2$ as follows: **[3]**

$$S_1 : \hat{e}_1 \to \frac{\hat{e}_1 - \hat{e}_2}{2},$$
$$S_1 : \hat{e}_2 \to \frac{\hat{e}_1 + 3\hat{e}_2}{2}$$

(i) Consider a rectangle under the action of this transformation. Describe physically what's happening here, and calculate the area from your diagram.

(ii) Obtain the matrix representing $S_1$, and show that its determinant is equal to the calculated area.

## 3 Programming

(a) Write a code which asks the user to input the $x$ and $y$ components of a vector in $\mathbb{R}^2$. **[1]**

(b) Define a function `rotate_vector`, which accepts two arguments – a vector `v` and an angle `theta` – and returns the original vector `v` rotated by the angle `theta`. **[3]**

(c) Define a function `vector_length` which accepts one argument – a vector `v` – and returns its length. Check that the rotated vector and the original vector have the same length. **[1]**