Assignment 5: Singular Matrices and Change of Basis

Due: March 6, 2020 (Friday) **Marks: 15**

1 The Inverse

(a) Let A and B be $n \times n$ real matrices. Assume that $A \neq B$, $A^3 = B^3$, and $A^2B = B^2A$. Is the matrix $A^2 + B^2$ invertible? **[2]**

(b) Let A be an invertible matrix. Assume that $A = A^{-1}$. What are the possible values for $\det(A)$? **[1]**

(c) Show that for a 2×2 matrix given below **[1]**

$$F = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

Show that the matrix representing F^{-1} is given by

$$F^{-1} = \frac{1}{\det(F)} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix},$$

2 Matrix Representations of Linear Maps

Consider a set we discussed in a previous assignment, $P_3(x)$, which is the space of all polynomials of degree at most 3. We saw that this set, along with the "normal" rules of addition of polynomials and multiplication by scalars was a vector space. In the following, we will take $p(x)$ to be an arbitrary element of $P_3(x)$.

(a) Find the dimension n of this vector space, and choose a natural basis. Show that there is a linear map L between $P_3(x)$ and $\mathbb{R}^n$, and find $L\big(p(x)\big)$. **[1]**

(b) Now consider the following transformation D that acts on the vectors of $P_3(x)$:

$$p(x) \xrightarrow{D} \frac{\mathrm{d}p}{\mathrm{d}x}$$

(i) Show that D can be expressed as a linear map from $P_3(x) \to P_3(x)$. **[1]**

(ii) Using L as defined above, calculate the matrix that represents D in $\mathbb{R}^n$, $\mathrm{Mat}_L(D)$. **[2]**

(iii) Find the Kernel of D. Is D invertible or singular? Answer this question using the properties of $\text{Mat}_L(D)$. **[1]**

(iv) Show that: **[1]**

$$\text{Dim}\Big(\ker(D)\Big) + \text{Dim}\Big(\text{Im}(D)\Big) = n.$$

(c) Choose a suitably **different** basis for $P_3(x)$. Define another linear map M that takes you from $P_3(x) \to \mathbb{R}^n$.

(i) Find the matrix B as defined in class: **[1]**

$$M = BL$$

(ii) Find $\text{Mat}_M(D)$, and show that **[2]**

$$\text{Mat}_M(D) = B\,\text{Mat}_L(D)\,B^{-1}$$

3 Computation

We've spoken about nilpotent matrices in the lectures, but we restricted ourselves to 2×2 matrices N, such that $N \neq 0$, but $N^2 = 0$. Let us now generalise this to n dimensions, and let P be some $n \times n$ matrix, such that $P^k = 0$ (where $k < n$ is called the *degree* of the nilpotent matrix). For example, consider the matrix

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies X^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad Y = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies Y^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies Y^3 = 0.$$

X and Y have degrees 2 and 3, respectively.

(a) You are given the following matrix

$$P = \begin{pmatrix} 2 & 2 & 2 & -3 \\ 6 & 1 & 1 & -4 \\ 1 & 6 & 1 & -4 \\ 1 & 1 & 6 & -4 \end{pmatrix}$$

Write a python function `nilpotent_degree` which accepts one argument, a matrix `P`, and multiplies the matrix with with itself until all the resulting entries are zero. The function then returns the integer `k`. Find k for P. **[2]**