Assignment 7: Eigenvalues, Eigenvectors, and Markov Processes

Due: April 2, 2020 (Thursday) **Marks: 15**

1 Epidemics: Markov Processes and the *SIR* Model

Given the current medical situation regarding the novel coronavirus (COVID-19), it seems pertinent to show how we can apply the "abstract" ideas that we learn in this course to a more tangible problem, specifically the study of epidemics.

We will be studying a very simplified model of epidemics known as the *SIR model*, where *SIR* stands for **S**usceptible, **I**nfected, **R**ecovered which are the three classes the population is divided into: People who can get the illness (S), people who are infected (I), and people who have recovered (R) and therefore do not risk contracting the illness again. We make some simplifying assumptions:

- The population is a constant N: all people are either S, I, or R. Of course, this seems odd since people do die, but for the time being we will consider them part of R (since the dead certainly don't risk getting infected again!).
- We consider a "time-step" to be a day. We represent the "state" of a system to be the number of susceptible, number of infected, and number of recovered individuals. Thus, the state of the system on day 3 would be given by:

$$\psi_3 = \begin{pmatrix} S_3 \\ I_3 \\ R_3 \end{pmatrix}$$

At the end of every day, there is a fraction β of susceptible (or uninfected) people that become infected, and a probability γ that an infected person can recover.

(a) Using the above information, show that the matrix M that takes you from step n to step $n+1$. In other words, show that the matrix M such that $\psi_{n+1} = M \cdot \psi_n$. is given by: **[1]**

$$M = \begin{pmatrix} 1-\beta & 0 & 0 \\ \beta & 1-\gamma & 0 \\ 0 & \gamma & 1 \end{pmatrix}.$$

Check your answer by adding up the columns of M. What should the sum be equal to? Why?

(b) Find the eigenvalues and eigenvectors of M

(i) Manually, **[3]**

(ii) Using the SymPy package. **[3]**

(c) Diagonalise M, i.e. find the *change-of-basis* matrix B such that $M = BDB^{-1}$. **[1]**

(d) Suppose you initially have a total population of 100 individuals with 1 infected person. What is ψ_0? Show that **[2]**

$$\psi_n = M^n \psi_0.$$

(e) Calculate ψ_n by calculating M^n:

(i) Manually, **[2]**

(ii) Using the SymPy package. **[2]**

(f) Compute

$$\lim_{n\to\infty} \psi_n = \begin{pmatrix} S_\infty \\ I_\infty \\ R_\infty \end{pmatrix}$$

Is this what you would expect to happen? Why? **[1]**

Hint: Your answer won't depend on α or β since they are both probabilities and therefore $\alpha < 1$ and $\beta < 1$.

2 Bonus: The *SIRD* Model 15 Marks

(a) Do the same thing as above, but now let's say that an infected person still has a probability γ of recovering, but also has a probability δ of dying. You may do this entirely using SymPy.

Hint: The state of the system now is defined by **four** quantities: S, I, R, and D. **[10]**

(b) Using the same initial conditions as before, plot S_n, I_n, R_n, and D_n vs. n for different (interesting) values of the parameters β, γ, and δ. **[5]**