Assignment 9: Partial Derivatives

Due: April 16, 2020 (Thursday) **Marks: 15**

1 Partial Derivatives and Continuity

Often in Physics if a function is sufficiently smooth then we find that there is no difference between first differentiating keeping x constant and then y, or vice versa. In other words, if $f(x, y)$ is a function of two variables, we will generally assume that the functions we work with in physics are sufficiently smooth and satisfy

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

However, it is good to remember that there are some treacherous functions that may "look" smooth, but are not. For example, consider the function:

$$f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x, y) = (0,0), \\ 0, & (x, y) = (0,0), \end{cases}$$

(a) **Computation:** Plot the above two-variable function in a 3D-plot, and show that it looks smooth at $(x, y) = (0,0)$. (This site has a very short introduction to making 3D plots.) **[5]**

(b) Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and show that they're defined everywhere in the $xy-$plane. **[2]**

(c) Calculate $\left.\frac{\partial f}{\partial y}\right|_{(x,0)}$ which is the value of $\frac{\partial f}{\partial y}$ along the $x-$axis. **[2]**

(d) The above function is now a function of x alone. Calculate its derivative with respect to x $\left(\text{which we will call } \left.\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\right|_{(0,0)}\right)$. **[2]**

(e) Calculate $\left.\frac{\partial f}{\partial x}\right|_{(0,y)}$ which is the value of $\frac{\partial f}{\partial x}$ along the $y-$axis, and $\left.\frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right|_{(0,0)}$. **[2]**

(f) Show that **[2]**

$$\left.\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\right|_{(0,0)} \neq \left.\frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right|_{(0,0)}$$

Thus the second order "mixed" derivatives are not continuous at $(0,0)$. Physicists tend to be cavalier and assume that their functions are always sufficiently smooth to permit such exchange operations. The danger with this approach is that such cases may go unnoticed and errors may ensue.