DS 2: Dimensional Analysis

Philip Cherian

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1 Dimensional Analysis

Dimensional analysis is a powerful tool to extract relevant information from a physical system. Its purpose is to give certain information about the *relations* which hold between the measurable quantities associated with various physical phenomena. This method has the advantage of being rapid: it enables us to dispense with making a complete analysis of the physical system before drawing conclusions. On the other hand, it does not give us as complete information as might be obtained by carrying out a more detailed analysis.

1.1 Dimensions

A physical quantity that may be measured is usually[1] measured with respect to some standard. If the length of an object – say, a table – is to be measured, it is measured using a scale. This scale would say that the table measured, for example,

$$\text{Length of table } = 2 \text{ metres}$$

This is shorthand for saying that if two standard metre scales of 1 metre each were placed of after the other, they would have the same length as that of the table. Thus,

$$\text{Length of table } = \underbrace{2}_{\text{magnitude}} \times \underbrace{1 \text{ metre}}_{\text{unit}}$$

Similarly, if a duration of time is to be measured to be, say, five years,

$$\begin{aligned} \text{Duration } &= \underbrace{1}_{\text{magnitude}} \times \underbrace{1 \text{ year}}_{\text{unit}} \\ &= \underbrace{3 \times 10^7}_{\text{magnitude}} \times \underbrace{1 \text{ second}}_{\text{unit}} \end{aligned}$$

From the above example, it should be clear that the **magnitude** of a physical quantity depends on the unit chosen. If we naively only paid attention to the magnitude, a year might seem like a very small amount of time (just "1" unit), or a very large amount of time (30,000,000 units!).

[1] Though not always!

It is important here to draw a distinction between dimensions (which are physical quantities that may be measured) and units, which are the standards with reference to which they are measured.

For example, length is a dimension, but it may be measured using different units, the yard, the kilometre, the foot, or the light-year.

The physical quantities with dimension that we may measure can be broadly classified into two groups: **primary** and **secondary** quantities. Primary quantities are considered fundamental and irreducible (they cannot be written in terms of other quantities), while secondary quantities may be constructed from combinations primary quantities. The primary quantities that we will refer to through this course are mass (M), length (L), and time (T). An example here would perhaps be enlightening: consider the quantity "speed" or "velocity". It is defined to be the change in distance (Δx undergone by an object in a time interval Δt). i.e.

$$v = \frac{\Delta x}{\Delta t}$$

In terms of dimensions, the numerator of the above equation is a measure of length (and hence has dimension [L]), while the denominator is a measure of time (and hence has dimension [T]). The dimensions of velocity (usually represented by [v]), are given by

$$[v] = LT^{-1}.$$

Question: Show that the dimension of acceleration is

$$[a] = LT^{-2}$$

Question: What are the dimensions of force?

1.2 Principles

Let us begin with two basic principles on which physics is based:

Principle 1: Only magnitudes of quantities of similar dimension can be compared.

This should be quite obvious: it makes no sense to say that an object which is 10 m long is "smaller" than 1000 seconds. Similarly, questions like "Is a kilogram larger than a second", or "How many seconds are there in a metre", or "How long is a gram" don't make any sense.

Principle 2: Physical phenomena and physical laws do not depend on the unit system selected.

This may not be quite so intuitive, but is nevertheless one of the guiding principles of Physics. The units that we have chosen to describe the world, as described in the first lecture, are very personal units, which make sense to us as humans, but would seem very strange to a bug or an elephant.

The very nature of physics as a discipline requires that human beings and our choice of "standards" (such as the kilogram, or the metre, or the second) not be crucial to our description of the universe.[2]

The first principle should make it clear that you cannot add two quantities that have different dimensions (you can't add a kilogram to a metre). The second is far deeper: it means that we could change our units of measurement (say from Imperial to metric, or from human to bug) and our description of Nature would continue to be as valid.

1.3 Constants of Nature

This is particularly important as in Nature we have certain physical constants which have dimension (in other words, they are not merely magnitudes, but are measured in units). Whereas the physical quantity indicated by a physical constant does not depend on the unit system used to express the quantity, the numerical values of dimensional physical constants do depend on choice of unit system.

> The term "physical constant" refers to the physical quantity, and not to the numerical value within any given system of units.

One example is the speed of light $c = 3 \times 10^8$ m/s. As can be seen by the "m/s", the magnitude (3×10^8) of the speed of light depends on the units chosen. If you had chosen to measure it in cm/s, then it would be $c = 3 \times 10^{10}$ cm/s.

If you had chosen to measure it in cm/hr, it would be

$$c = 1.8 \times 10^{12} \text{ cm/hr} \tag{1}$$

1.4 Analysing dimensions

In the above example, the speed of light is measured in units of (say) cm/hr. We can check whether this is **dimensionally consistent**. We know that by definition, speed is a rate of change of distance with respect to time, and so it must have dimension

$$[c] = LT^{-1}$$

and on the right hand side, it is measured in units of cm ([cm] = L) over hour ([hour]=T), and so [cm/hr] = LT^{-1}. Thus, looking at the dimensions on either side of Equation (1),

$$\begin{aligned} c &= 1.8 \times 10^{12} \text{ cm/hr} \\ [c] &= [\text{cm/hr}] \\ LT^{-1} &= LT^{-1}. \end{aligned}$$

> Notice that in the second line above, the magnitude (1.8×10^{12}) suddenly disappears, but we still have an equality sign. This is because when we use the "[]" notation we are only concerned with the dimensions on either side, which is not affected by the magnitude (a pure number).

[2] This has sometimes been called the *Copernican Principle*, after Copernicus who argued that the universe – quite literally – did not revolve around us.

1.5 Method

Let us begin with an example. Suppose you want to find the maximum height $h_{\max}$ that a ball thrown upward with some speed u can reach. You spend some time thinking and decide that it could depend on the speed u (if u were greater, so would the height it reaches), the acceleration due to the gravity g (if this experiment were done on the moon, the ball would certainly move higher), and you feel that it would also depend on the mass m of the ball.

$h_{\max}$ should thus be some combination of u, g, and m. Since these quantities can't just be added (why not?), we need to combine them in some suitable way to get a quantity with dimension length. So we say

$$
\begin{aligned}
h_{\max} &\sim \underbrace{u \times u \times u \times \ldots}_{a \text{ times}} \times \underbrace{g \times g \times g \times \ldots}_{b \text{ times}} \times \underbrace{m \times m \times m \times \ldots}_{c \text{ times}} \\
&\sim u^a g^b m^c
\end{aligned}
$$

It's important to realise that we have used the ∼ symbol on purpose. This method of analysis (as we saw in the last section) cannot tell us anything about the dimensionless number in front of these units (like the magnitude 1.8×10^{12} in our last example). Thus, our answer is only very roughly correct, and there will most often be a constant which is larger than 1 in front of it.

Now, we consider only the dimensions on either side.

$$
\begin{aligned}
&= [u]^a [g]^b [m]^c \\
L &= (LT^{-1})^a (LT^{-2})^b (M)^c \\
L^1 &= L^{a+b} T^{-(a+2b)} M^c \\
M^0 L^1 T^0 &= L^{a+b} T^{-(a+2b)} M^c
\end{aligned}
$$

Now, since we're dealing with dimensions, the equality ("=") sign has replaced the ∼ sign.

In the last step, we compare the dimensions on either side of the equation. The left-hand side, which only has length, has no mass or time, and we represent this by placing mass and time to the power 0.[3] We then compare the powers on either side. Since there is no mass on the left-hand side, this means that on the right-hand side, $c = 0$.

Similarly, since there is no time on the left-hand side, the power of time on the right-hand side should also be zero.

$$\implies -(a+2b) = 0 \quad \implies a = -2b$$

And last of all, since there is only one length on the left-hand side, the power of length on the right-hand side must be 1.

$$a + b = 1$$

[3] Since any number to the power 0 is 1, which is dimensionless.

Question: Show that this implies that

$$h_{\text{max}} \sim \frac{u^2}{g}$$

Doing a more detailed analysis (which you should all be able to do), you will find that

$$h_{\text{max}} = \frac{1}{2}\frac{u^2}{g}$$

which is not far off!

From this, we also reach what might seem to be a strange conclusion: while we had assumed that mass might be a factor, it turns out that the mass does not seem to contribute to the maximum height reached! The reason for this is that **there is no way in this scheme of things, for mass to be placed in the above equation!**

The method used in dimensional analysis is simple. However, using it in more complicated situation becomes an art. Suppose you have a physical situation, and you want to decide quantitatively how some parameter depends on the other parameters of the problem. Here are the steps:

(a) **Step 1:** Find the relevant parameters that the problem depends on. In the above example, it would be u, g, and m.

(b) **Step 2:** Write the quantity that you are interested in as some product of powers of the other quantities.

(c) **Step 3:** Expand these quantities in terms of their fundamental dimensions ($u = LT^{-1}$, $g = LT^{-2}$, etc.) and equate the dimensions on either side, deriving a relation between the powers.

(d) **Step 4:** Solve for the powers to get the final relation. Interpret your result (i.e. the mass does not affect the maximum height, etc.)

Obviously, the most important step is the first: finding the quantities that matter, and this is no small task. It requires a slight understanding of physics.

2 Length and time scales in differential equations

(a) Consider the differential equation for simple harmonic motion.

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -kx.$$

Identify a quantity of dimension time (which we will call τ) using the constants of this problem. In other words, find some combination of the constants given in this problem that results in a quantity that has dimensions of time.

(b) Now let's assume that we measure time in units of τ. i.e., instead of "some number of seconds have passed", we say "some number of τs have passed". Show that this just means that you're shifting from the variable t to the variable $\eta = t/\tau$. What are the dimensions of η?

(c) Rewrite the differential equation in terms of η and x. You should be able to see that the problem no longer depends on any parameters that have units. This is known as "nondimensionalising" a differential equation.[4] Can you think of any benefits of doing this?

(d) Now consider the equation for a Duffing oscillator. Such a system is used to model "stiff springs":

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - \alpha x^3.$$

In this case, you can define *two*: one with dimensions time that we'll call τ, and one with dimensions of length that we'll call ℓ.

(e) What do τ and κ represent physically? Now let's count time in units of τ, and length in units of ℓ. Define two new variables $\eta = t/\tau$, and $\lambda = x/\ell$.

(f) Rewrite the differential equation in terms of η and λ, just as before. Show that the differential equation can now be written as:

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}\eta^2} = -\left(\lambda + \lambda^3\right).$$

Think about what happens when $x \ll \ell$ and $x \gg \ell$ (i.e., when $\lambda \ll 1$ and $\lambda \gg 1$ respectively). What is this saying "physically"?

[4] See more about this on Wikipedia here: https://en.wikipedia.org/wiki/Nondimensionalization.