DS 4: An Introduction to Vector Spaces

Philip Cherian

February 6, 2020

1 Axioms of Vector Spaces

(a) Suppose V is a vector space. (i) Show that V cannot have more than one additive identity. (ii) Show that every element v in V has only one additive inverse.

(b) In each of the cases below, a set V is provided, equipped with specific operations. In each case, determine whether V is a vector space. If it isn't, state the axioms that V violates. In each case, identify the **additive identity**.

(i) $V = \left\{\text{Solutions to the simple harmonic oscillator}\right\}$, equipped with the usual definition of addition of functions and their multiplication by scalars.

(ii) $V = \left\{n \times n \text{ matrices with positive entries}\right\}$, equipped with standard matrix operations.

(iii) $V = \left\{n \times n \text{ real symmetric matrices}\right\}$, equipped with standard matrix operations.

(iv) $V = \left\{n \times n \text{ diagonal matrices}\right\}$, equipped with standard matrix operations.

(v) $V = \left\{\text{Functions defined for all } x, \text{ with } f(0) = 0\right\}$, equipped with the standard operations for addition of functions and their multiplication by scalars.

(vi) $V = \left\{\text{Functions defined for all } x, \text{ with } f(0) = 1\right\}$, equipped with the standard operations for addition of functions and their multiplication by scalars.

(c) Show that the set $\mathbb{R}^+$ of positive real numbers is a vector space when the action of "addition" $x \oplus y$ is interpreted to mean the product of x and y (so that $2 \oplus 3 = 6$), and "muliplication" by a scalar is defined by $r \otimes x = x^r$.

(d) If in the question above, we were dealing with $\mathbb{R}^-$ (negative real numbers) would it still be a vector space? If yes, show that all the axioms are satisfied. If no, explain why not.

(e) Consider the set of all Fibonacci sequences. A Fibonacci sequence is one in which each number is the sum of the two preceding numbers. **Note: each element** of this set is a sequence. The first two elements of the Fibonacci sequence can be chosen arbitrarily, and give rise to different sequences. For example, below are some sequences parametrised by their first two elements:

$$f_{01} = \left\{0, 1, 1, 2, 3, 5, \dots\right\}$$
$$f_{22} = \left\{2, 2, 4, 6, 10, 16, \dots\right\}$$
$$\vdots$$