DS 7: Linear Maps as Matrices

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1 The Inverse

(a) Let A be an $n \times n$ matrix of real numbers. Which of the following statements are equivalent to: "the matrix A is invertible"? Explain *briefly*.

(i) The columns of A are linearly independent.

(ii) The columns of A span $\mathbb{R}^n$.

(iii) The rows of A are linearly independent.

(iv) The kernel of A is 0.

(v) The only solution of the homogeneous equations $Ax = 0$ is $x = 0$.

(vi) The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by A is one-to-one.

(vii) The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by A is onto.

(viii) $\det(A) \neq 0$.

2 Singular Matrices

(a) Consider a transformation which maps a three-dimensional vector to its projection on the x axis.

(i) Show that this transformation is linear.

(ii) Determine the matrix for this transformation, P_x.

(iii) Show that $P_x \cdot P_x = P_x$.

(iv) Find the image of this transformation. What is its dimension?

(v) Find the kernel of this transformation. What is its dimension?

(b) Now consider a transformation which maps a three-dimensional vector to its projection on the $x-$ $-y$ plane, and repeat the above process.

3 Linear Transformations as Matrices

Consider the set of all solutions to the differential equation

$$y''(x) + y(x) = 0$$

.

(a) Show that this set is a vector space under the usual operations of addition and multiplication by real numbers.

(b) Find the dimension n of this vector space.

(c) Choose a basis $\{\hat{u}_1, \hat{u}_2, \dots \hat{u}_n\}$ for this set, and show that there is a map (L) between this space and $\mathbb{R}^n$, such that

$$L(\hat{u}_1) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$L(\hat{u}_2) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$\vdots \quad \text{etc.}$$

(d) Let's now choose some "abstract" operation S, shifting our variable x by $\pi/2$. i.e. $x \to x + \pi/2$.

(i) How does S act on the basis vectors $\hat{u}_1, \hat{u}_2, \dots$?

(ii) In this new basis, we now have a new map M. Find the matrix representing the transformations S in $\mathbb{R}^n$ for the map M.

(iii) Find the matrix B which allows us to go from L to M,

$$M = BL.$$

(iv) Show that

$$\text{Mat}_M\Big(S\Big) = B\,\text{Mat}_L\Big(S\Big)\,B^{-1}.$$