DS 8: Eigenvalues, Eigenvectors, and Markov Processes

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1 Eigenvalues and Eigenvectors

(a) Show that if

$$M = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

then

$$M^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

(b) Consider the matrix

$$F = \begin{pmatrix} -7 & 18 \\ -3 & 8 \end{pmatrix},$$

and diagonalise it as follows:

(i) Find the characteristic polynomial $P(\lambda)$ and set it equal to 0 to find the eigenvalues of F.

(ii) Check that $P(F) = 0$, i.e. that F satisfies the Cayley-Hamilton theorem.

(iii) Find an eigenvector for each eigenvalue (you may choose the $y-$component to be 1 for both of them), and compute the "diagonal" matrix Λ, i.e. the matrix F in its eigenvalue basis.

(iv) Calculate the change of basis matrix B, and confirm that $F = B\Lambda B^{-1}$.

(c) Consider the two-dimensional rotation matrix $R(\theta)$.

(i) What would it mean for it to have eigenvectors?

(ii) Calculate its characteristic polynomial.

(iii) For which value(s) of θ does the characteristic polynomial have roots? What is $R(\theta)$ in these cases? What are the corresponding eigenvectors?

(d) Let's solve a problem that's given in the Lecture session. Consider a matrix T

$$\begin{pmatrix} 1 - T_{21} & T_{12} \\ T_{21} & 1 - T_{12} \end{pmatrix}$$

(i) Find its eigenvalues and eigenvectors,

(ii) Find the "change of basis" matrix, B.

2 Markov Processes

(a) Let's consider the Fibonacci Series, defined by the following *difference* equation:

$$x_{n+2} = x_{n+1} + x_n.$$

We need to provide two initial pieces of information to totally specify the series, so let's choose $x_0 = 0, x_1 = 1$. From here on, all the remaining elements are completely specified.

(i) Find the limit

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

(ii) Find a matrix A that allows you to go from

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}.$$

(iii) Using the above result, show that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists, and compute its value.

(iv) Are there any values of x_0 and x_1 that give a *different* result for this limit?