DS 9: Orthogonal Matrices and Markov Processes using SymPy

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1 Eigenvalues and Eigenvectors

(a) Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

(b) Show that A^T and A have the same eigenvalues. What about their eigenvectors?

(c) If A is idempotent ($A^2 = A$), show that its eigenvalues are 0 or 1. If A is nilpotent ($A^k = 0$), show that all eigenvalues are 0.

2 Orthogonal Matrices

A matrix O is said to be **orthogonal** if $O^T O = \mathbb{1}$.

(a) Suppose A is orthogonal. Find the determinant of A.

(b) Find a general representation of orthogonal matrices. Use this to find its eigenvalues and eigenvectors. Show that the eigenvalues λ_i satisfy the condition $|\lambda_i| = 1$.

(c) If A and B are orthogonal matrices, is their produce AB orthogonal?

3 Epidemics: the SIS model

Let's consider a simple model of diseases where the population N is fixed and divided into two groups, the susceptible (S) and the infected (I). The susceptible become infected at a constant rate r_1, meaning that at the end of each day, there is a probability of r_1 that an uninfected person can become infected. Similarly, people who are infected may overcome the disease and become susceptible to infection again at a rate r_2. Let's imagine that we initially begin with 1 infected person and $N - 1$ susceptibles.

(a) How would you represent the state of the system ψ_n?

(b) Find the transition matrix M. i.e. find the matrix M such that $\psi_{n+1} = M \cdot \psi_n$.

(c) Calculate M^n, and use this to find ψ_∞.

(d) Write a code to compute ψ_n at different values of n, and plot the number of susceptible and infected individuals in the population as a function of time.