DS 10: Coupled Differential Equations

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1 Solving Differential Equations without Matrices

1.1 Why exponentials?

Exponentials have a special part to play in the solutions to the sort of differential equations you will see in this course, and this is because they can (in many cases) transform a differential equation into an *algebraic* equation that you know how to solve.

For example, consider the following differential equation that you are all (hopefully) familiar with:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega_0^2 x = 0.$$

Suppose we look for a solution of the form $Ae^{\alpha t}$. The reason we do this is just glorified guesswork. Such a solution – when plugged into the above equation – will give us:

$$(\alpha^2 + \omega_0^2) Ae^{\alpha t} = 0,$$

and since A and $e^{\alpha t}$ are never zero, this means that $\alpha = \pm i\omega_0$.

Thus, we have two functions which solve the differential equation:

$$\begin{aligned} x_1(t) &= Ae^{i\omega_0 t} \\ x_2(t) &= Ae^{-i\omega_0 t} \end{aligned} \tag{1}$$

Now, we know that the solutions to a nth order differential equation form a vector space of dimension n, and to describe any vector space of dimension n we need at most 2 linearly independent vectors.

Exercise: Show that $x_1(t)$ and $x_2(t)$ are linearly independent.

Thus, the general solution can be written as

$$x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t},$$

where A and B will be fixed by the initial conditions.

1.2 Why complex numbers?

Some of you might be surprised at why we introduce complex numbers to solve simple physical situations which deal with "real" physical quantities. It's a good idea to examine why we do this so that you know why (and therefore, when) it is advisable to use this "trick".

The first step is to realise that there are two complex notations that are equivalent. The first, with which most of you must be familiar from school, is that a complex number z can be denoted by $x + iy$, where x and y are both real. This is the "Cartesian" representation of complex numbers, but – just as with a normal xy-plane, one could just as well represent these points in *polar* form. In other words, we could imagine z being represented by some "distance" from the origin r, and some angle from the x-axis, θ. Clearly,

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta & \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned} \tag{2}$$

Substituting these results into z, we can see that

$$z = x + iy = r\cos\theta + r\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{3}$$

Where in the last step we've used an important mathematical formula called Euler's Formula. The standard way to prove this is using Taylor series (which we will see later in this course) but for now, a short proof will be sketched out in the exercise below.

Exercise: Begin by defining $f(\theta) = \cos\theta + i\sin\theta$. We are going to show that $f(\theta) = e^{i\theta}$ for all values of θ.

(a) Show that $f'(\theta) = if(\theta)$ for all θ. (**Hint:** Use the fact that $\frac{1}{i} = -i$.)

(b) By solving the above differential equation, show that $f(\theta) = e^{i\theta}$ for all θ. (Remember, a first order differential equation has one arbitrary constant, which can be set by $f(0)$.)

It is this property of a complex number (that it can be written as a real number times a complex exponential) that makes it invaluable to solving linear (homogeneous[1]) differential equations.

Exercise: In particular, looking at the general solution in Equation (1.1), use the Euler formula to show that

$$x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t).$$

and find C_1 and C_2 in terms of A and B.

Important Note: Complex exponentials behave quite differently from real exponentials. Suppose ω is a real number. Then $e^{\omega t}$ is a continuously increasing (or decreasing, depending on the sign of ω) function in time. However, $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ is an *oscillatory* function! Thus, the appearance of complex exponentials highlights **oscillatory behaviour**!

[1] This word might be unfamiliar, and will become clearer next year in your MP course. Suffice it to say that it can be used to solve differential equations of the form $a_n x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \ldots a_1 x'(t) + a_0 x(t) = 0$.

2 Solving Differential Equations with Matrices

2.1 Two uncoupled masses

Let's start off easy: consider the simple system given in Figure (1)

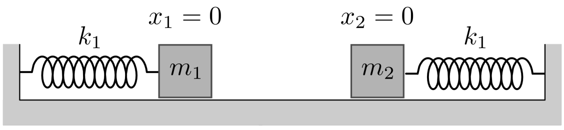

Figure 1: Two masses are allowed to oscillate, and neither is affected by the other.

The state of this system is completely specified by $x_1(t)$ and $x_2(t)$.

Exercise: Let's convert this into a matrix problem:

(a) Begin by showing that $x_1(t)$ and $x_2(t)$ satisfy the following set of differential equations:

$$m_1\ddot{x}_1 = -k_1 x_1$$
$$m_2\ddot{x}_2 = -k_1 x_2$$

(b) Show that this can be converted into the following matrix equation:

$$\underbrace{\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}}_{\underline{\underline{M}}} \underbrace{\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}}_{\underline{\ddot{X}}} = \underbrace{-\begin{pmatrix} k_1 & 0 \\ 0 & k_1 \end{pmatrix}}_{-\underline{\underline{K}}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\underline{X}}$$

(c) Show that the above equation can be written as shown below,

$$\ddot{\mathbf{X}} = -\left(M^{-1}K\right)\mathbf{X} = -\Lambda\mathbf{X} = -\begin{pmatrix} \frac{k_1}{m_1} & 0 \\ 0 & \frac{k_1}{m_2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Thus, in matrix form, the pair of equations seems to resemble a *single* (matrix) equation:[2]

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{X} = -\Lambda\mathbf{X} \tag{4}$$

We can solve such an equation in exactly the same manner as we did for the "normal" differential equations, but with one added complication. Let us "guess" a solution[3] of the form

$$\mathbf{X}(t) = \mathbf{C}_\alpha e^{i\alpha t},$$

where $\mathbf{C}_\alpha$ is a *constant* column-vector that we need to find (which may be different for different αs), and α is a real number.

[2] I'll ignore the underbars from now on, capital letters will denote matrices like K and M, and bold-capitals will denote vectors, like $\mathbf{X}$.

[3] From time to time, we will call this an "ansatz", which is just a fancy way of saying it's a guess.

Exercise: Substitute the above ansatz into Equation (4).

(a) Show that $\alpha = \pm\omega_1, \pm\omega_2$.

(b) If this is the case, show that Equation (4) reduces to

$$\Lambda \mathbf{C}_\alpha = \alpha^2 \mathbf{C}_\alpha \tag{5}$$

From the above exercise it must be clear that the C_α must be the **eigenvectors** of Λ corresponding to the eigenvalues α^2. We had earlier seen that there were four distinct values of α, but only *two* distinct values of α^2. We thus have to solve two eigenvalue equations:

$$\Lambda \mathbf{C}_{\pm\omega_1} = \omega_1^2 \mathbf{C}_{\pm\omega_1}$$
$$\Lambda \mathbf{C}_{\pm\omega_2} = \omega_2^2 \mathbf{C}_{\pm\omega_2}$$

Exercise: Solve the above equations and show that

$$\mathbf{C}_{+\omega_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{C}_{-\omega_1},$$
$$\mathbf{C}_{+\omega_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{C}_{-\omega_2},$$

Hint: This should be trivial, since the matrix Λ is diagonal!

The different vectors that satisfy this eigenvalue equation are known as the **Normal Modes** of the system, and their corresponding eigenvalues are known as the **normal mode frequencies**.

Important Note: (Just to make sure you don't forget)

(a) The eigenvectors of the Λ matrix are the **normal modes**, a special linearly independent basis that is the "physical" basis of our problem.

(b) The eigenvalues of the Λ matrix are the **normal mode frequencies**, since the solutions change in time according to these frequencies.

The general solution to this differential equation is thus given by:

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = D_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\omega_1 t} + D_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\omega_1 t} + D_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\omega_2 t} + D_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\omega_2 t} = \begin{pmatrix} D_1 e^{i\omega_1 t} + D_2 e^{-i\omega_1 t} \\ D_3 e^{i\omega_2 t} + D_4 e^{-i\omega_2 t} \end{pmatrix} \tag{6}$$

Note that even though (for example) $\pm\omega_1$ have the same eigenvectors (or normal modes), they each have a different exponential term. The resulting solution is just that for two independent harmonic oscillators, and the different constants D_i will be determined by the initial conditions.

The reason this problem was simple (nay, trivial) was because the matrix Λ was already diagonal. The reason the matrix Λ was diagonal was because the two masses did not "talk" to each other. In other words, they weren't "coupled" to each other. We will now allow for some coupling by introducing a spring with some new spring constant k_2 in between them. Now, the force on the mass m_1 will *also* depend on the position of m_2, and vice versa.

2.2 Two coupled masses

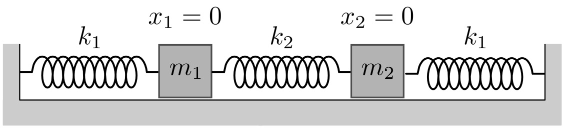

Figure 2: Two masses are allowed to oscillate as before, but are further coupled by a spring k_2.

Exercise: Choose $m_1 = m = m_2$ and repeat the above procedure in the previous subsection to this problem. Find the Λ matrix such that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{X} = -\Lambda\mathbf{X}.$$

(a) Guess a solution of the form $\mathbf{C}_\alpha e^{i\alpha t}$, and find the values of α^2 possible. (**Hint:** These are just the eigenvalues of Λ.) These are the **normal mode frequencies**. Show that they are

$$\omega_1^2 = \frac{k_1}{m} \quad \omega_2^2 = \frac{k_1 + 2k_2}{m}.$$

(b) Find the column-vectors $\mathbf{C}_\alpha$ that correspond to each eigenvalue α^2. These are the **normal modes**. You should find that

$$\omega_1^2 \text{ has eigenvector } \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \omega_2^2 \text{ has eigenvector } \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

If you've done the above calculations correctly, you should find the following:

$$\Lambda = \begin{pmatrix} \dfrac{k_1 + k_2}{m} & -\dfrac{k_2}{m} \\ -\dfrac{k_2}{m} & \dfrac{k_1 + k_2}{m} \end{pmatrix}$$

and the general solution $\mathbf{X}(t)$ is

$$\mathbf{X}(t) = \left(D_1 e^{i\omega_1 t} + D_2 e^{-i\omega_1 t}\right)\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} + \left(D_3 e^{i\omega_2 t} + D_4 e^{-i\omega_2 t}\right)\begin{pmatrix} \mathbf{1} \\ -\mathbf{1} \end{pmatrix}$$

where the D_is are determined by the initial conditions. As before, these four arbitrary constants are determined by the initial conditions $(x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0)$. Looking closer at the solution, you should be able to see that it can be written as

$$\mathbf{X}(t) = \text{Function with frequency } \omega_1 \times \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} + \text{Function with frequency } \omega_2 \times \begin{pmatrix} \mathbf{1} \\ -\mathbf{1} \end{pmatrix}$$

As we shall see, this is a characteristic feature of such equations.

2.3 General procedure to solve coupled differential equations

Armed with our two examples, we can now generalise the method to solve such systems of coupled differential equations.

Write out the differential equations Looking at the physical system, use Newton's law to write out the force on each mass (and therefore get n differential equations).

Convert it to matrix form Define an $n-$dimensional column vector $\mathbf{X}$, and write the above equations as a single matrix equation

$$\ddot{\mathbf{X}} = -\Lambda \mathbf{X}.$$

Find the eigenvalues of Λ These eigenvalues $\omega_1^2, \omega_2^2, \ldots, \omega_n^2$ are the **normal mode frequencies** (squared).

Find the eigenvectors of Λ Using the eigenvalues found, the eigenvectors can be obtained which are called the **normal modes**. These eigenvectors $\mathbf{v_i}$ represent combinations of $x_1(t), x_2(t), \ldots, x_n(t)$ which each oscillate at a specific frequency ω_i.

Write out the general solution The general solution is given by

$$\mathbf{X}(t) = \left(D_1 e^{i\omega_1 t} + D_2 e^{-i\omega_1 t}\right)\mathbf{v_1} + \left(D_3 e^{i\omega_2 t} + D_4 e^{-i\omega_2 t}\right)\mathbf{v_2} + \ldots + \left(D_{2n-1} e^{i\omega_n t} + D_{2n} e^{-i\omega_n t}\right)\mathbf{v_n}.$$

Determine the constants D_i The $2n$ constants are determined by setting the n initial positions of the masses, and their n initial velocities. Once this is done, the entire problem has been solved.

2.4 Example: Solving a harder system

Let's attempt to solve the system in Figure (3). The first step is to determine the differential equations that this system satisfies. Each mass is attached to two springs on either side, and so will have two restoring forces due to them. It's very important to get the signs of these terms correct, and here's a simple trick to do that: imagine a snapshot of the system, and choose $0 < x_1 < x_2 < x_3$ (i.e. all the masses are displaced in the same direction – to the right which we will take as being "positive" displacement). As we will show below, this makes it easy to write out the equations.

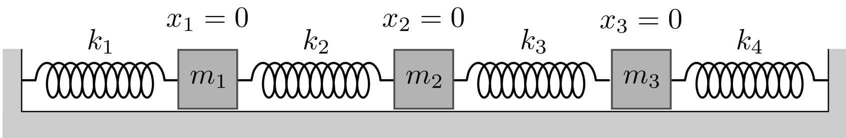

Figure 3: Three masses are connected by four springs on a frictionless surface.

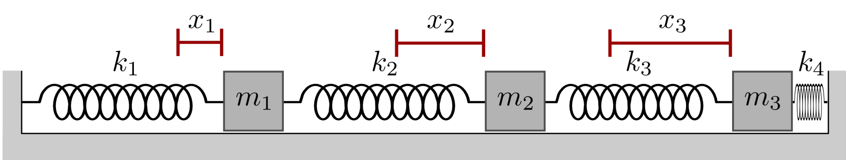

Figure 4: Consider a snapshot of each of the masses moved by a small amount. Without loss of generality, choose $0 < x_1 < x_2 < x_3$.

Mass 1 The force on m_1 is due to the springs k_1 and k_2. We have chosen $0 < x_1 < x_2$. Since x_1 is positive, the spring k_1 is extended to the right, so it tries to pull m_1 with a force $-kx_1$ (to the left).[4] In the case of the spring k_2, since $x_2 - x_1 > 0$, the spring has been extended. This spring would like to move m_1 to the *right* (it would *pull* m_1 towards its centre), and so the force due to it is $+k_2(x_2 - x_1)$. Thus

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1).$$

Mass 2 The force on m_2 is due to the springs k_2 and k_3, and we have chosen $0 < x_1 < x_2 < x_3$. We've seen that the spring k_2 is extended by a length $x_2 - x_1$, and so it would like to pull m_2 towards its centre, which is to the *left*. Thus, the force due to k_2 is $-k_2(x_2 - x_1)$. In the case of spring k_3, it has been extended by a length $x_3 - x_2 > 0$, and so it would like to pull m_2 towards *its* centre, i.e. to the right. Thus

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) + k_3(x_3 - x_2).$$

Mass 3 The force on m_3 is due to k_3 and k_4. As before, since $x_3 - x_2 > 0$, m_3 is pulled to the *left* with a force $-k_3(x_3 - x_2)$. It is also pushed to the left with a force $-k_4x_3$ by the last spring that has been compressed, so

$$m_3\ddot{x}_3 = -k_3(x_3 - x_2) - k_4x_3.$$

Exercise: Write the above system as a matrix equation, and show that the matrix Λ is given by

$$\Lambda = \begin{pmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} & 0 \\ -\frac{k_2}{m_2} & \frac{k_2+k_3}{m_2} & -\frac{k_3}{m_2} \\ 0 & -\frac{k_3}{m_3} & \frac{k_3+k_4}{m_3} \end{pmatrix}$$

We will now simplify the problem greatly by choosing $k_1 = k_2 = k_3 = k_4 = k$, and $m_1 = m_2 = m_3 = m$. We can also call $\omega_0^2 = k/m$. This greatly simplifies the matrix Λ to:

$$\Lambda = \omega^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

We can now find the normal mode frequencies, and the normal modes.

Exercise: Show that the eigenvalues of Λ are

$$\omega_1^2 = 2\omega_0^2$$
$$\omega_2^2 = (2 - \sqrt{2})\omega_0^2$$
$$\omega_3^2 = (2 + \sqrt{2})\omega_0^2$$

and find their corresponding eigenvectors. Use this to find the general solution.

[4] If we had chosen an x_1 that was to the left (i.e. negative), then this would still be true, since the net force would be positive, i.e. to the right.