DS 11:
Coupled Differential Equations

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1 Solving Differential Equations with Matrices

(a) Consider the following second-order differential equation $\ddot{x} + \omega_0^2 x = 0$.

  (i) Write this differential equation as two (coupled) first-order differential equations.

  (ii) Write out the two equations in matrix form. Define a matrix M that

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t) = M V(t).$$

  This is important as it shows us that we can write an nth order differential equation as n coupled first order differential equations.

  (iii) Show that in this case the matrix M has the nice property that $M^2 = -\mathbb{1}$.

  (iv) The formal solution to the above equation is $V(t) = e^{tM} V(0)$. The exponential of a matrix is defined by its series expansion:

$$e^{tM} = \mathbb{1} + tM + \frac{(tM)^2}{2!} + \frac{(tM)^3}{3!} \ldots = \sum_{n=0}^{\infty} \frac{(tM)^n}{n!}.$$

  Find e^{tM} in this simple case.

  (v) Find the solution to the given differential equation.

(b) Of course, the matrix M is a special matrix, and not all matrices have the nice property that $M^2 = -\mathbb{1}$. This is why we diagonalise matrices. However, there is another way to solve such problems. Consider the following equations:

$$\dot{x} = -2x + y$$
$$\dot{y} = x - 2y$$

  (i) Find the eigenvalues and eigenvectors of the matrix associated with these equations.

  (ii) We've already seen that the formal solution to this equation is $V(t) = e^{tM} V(0)$.

  (iii) Can you think of another way to use the eigenvalues and eigenvectors to solve this problem simply?

2 Computation

(a) Go through this short guide to 3D plotting[1] that will help you with this week's assignment.

[1] https://jakevdp.github.io/PythonDataScienceHandbook/04.12-three-dimensional-plotting.html