# DS 12:
# Partial Derivatives and the Jacobian

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## 1 The Jacobian

(a) Find the Jacobian for the following transformations:

  (i) $x = 4u - 3v^2 \qquad y = u^2 - 6v$

## 2 Coordinate Transformations

(a) Find the Jacobian Matrix $J$ and its determinant $\det J$ for the transformation from Cartesian to spherical coordinates.

(b) Find how the infinitesimal area element changes when you go from Cartesian to polar coordinates.

(c) We can use the above result to calculate a very important integral in physics, the Gaussian integral:

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2} \mathrm{d}x.$$

You may try to use the standard techniques of substitution and integration by parts and see that neither of them allow you to compute the integral simply. We will have to find another way to do this.

  (i) Convince yourself that you can write

$$I^2 = \int_{-\infty}^{\infty} e^{-\alpha x^2} \mathrm{d}x \int_{-\infty}^{\infty} e^{-\alpha y^2} \mathrm{d}y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\alpha (x^2+y^2)} \mathrm{d}x\mathrm{d}y$$

  (ii) This integral itself is quite hard to calculate as well, at least in the coordinates $x$ and $y$. The trick is to now move to *polar* coordinates. Show that in polar coordinates

$$I^2 = \int_{0}^{\infty} \int_{0}^{2\pi} e^{-\alpha r^2} r \mathrm{d}r\mathrm{d}\theta.$$

  Pay careful attention to the limits. How can you justify them?

  (iii) Show that this integral is much simpler to calculate, and hence deduce that

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2} \mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}.$$