DS 13: Multidimensional Integrals

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1 Multidimensional Integrals

1.1 A circular disk

(a) Convince yourself that the area of a circle of radius R in the $xy-$plane is given by:

$$A = \iint_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}y\, f(x,y) \qquad \text{where } f(x,y) = \begin{cases} 1, & x^2+y^2 \le R^2 \\ 0, & x^2+y^2 > R^2 \end{cases}$$

(b) Use the above definition to calculate the area of a circular disk using Cartesian coordinates.

(c) The symmetry of the problem suggests that we use polar coordinates. Convince yourself that in polar coordinates, the integral is given by changing

$$f(r,\theta) = \begin{cases} 1, & r \le R, \\ 0, & r > R \end{cases},$$

and calculate the integral using *these* coordinates.

(d) Suppose now you were interested in calculating the mass of such a disk. (For simplicity, let's assume the disk to be two dimensional, with a finite non-zero mass per unit area.) If the disk were uniform, the mass would just be the mass per unit area times the area. But what if the density depended on the position? A simple generalisation would be to have it depend only on the radial coordinate. i.e. the mass per unit area is given by $f(r,\theta) = Kr^2$. Calculate the mass of the disk.

1.2 A more general area

(a) Consider a more complicated integral of the form:

$$\iint_{\mathbb{R}^2} \mathrm{d}u\mathrm{d}v\; f\left(\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2\right)$$

Using the fact that the integrand depends on u and v only in a particular combination motivates the introduction of "generalised polar coordinates",

$$u = a\mu\cos\phi \qquad v = b\mu\sin\phi$$

$$\mu = \sqrt{\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2} \qquad \phi = \arctan(av/bu)$$

(a) Find the Jacobian of the above transformation.

(b) Find the function $f(\mu,\phi)$ which represents the area bounded by an ellipse with axes a and b. Use this to compute the area of the ellipse.