DS 14: The Taylor Series and the Stirling Approximation

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1 Planetary Orbits and Simple Harmonic Motion

This problem will help you understand the orbit of a planet around the Sun, without having to know much mechanics to do it. You will learn next year in your course on *Classical Mechanics* that the effective potential energy of a planet in orbit around the Sun is given by the following expression when the planet is much lighter than the Sun:

$$V_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{GMm}{r},$$

where L is the angular momentum, a conserved quantity, M and m are the masses of the Sun and planet respectively, and r is the radial distance of the planet from the Sun, which varies as the planet goes around, since the orbit is elliptical.

(a) What are the symmetries of the gravitational field of the Sun? What coordinate system is being used here? Why?

(b) Show that V_{eff} has a minimum, and find the point $r = r_0$, where it occurs. Do a dimensional check on the expression you get for r_0. Does the fact that V_{eff} has a minimum suggest stability or instability? Stability with respect to what?

(c) Plot V_{eff} for reasonable values of the constants and convince yourself that has one one minimum.

(d) Define a dimensionless effective potential energy by dividing V_{eff} by GMm/r_0, and a dimensionless radius by dividing r by r_0. Plot the dimensionless potential energy against the dimensionless radius.

(e) Do a Taylor Series expansion up to second order of this expression about the minimum. The radial motion predicted by your analysis should be oscillatory. Find the time period of the radial oscillation and compare it with the time period of a planet in a *circular* orbit at radius r_0. Try to make sense of all of this put together.

2 The Stirling Approximation

Factorials appear very often in physics (especially in Statistical Mechanics) but they can be quite difficult to work with since they very quickly become unmanageably large. One interesting result was discovered by Euler who noticed that for integers n,

$$\int_0^\infty \mathrm{d}u\, u^n e^{-u} = n! \quad (n = 1, 2, 3, \ldots).$$

The above integral is not only defined for integers. In general, one could replace n by some continuous variable z, and the resulting integral is called the Gamma Function, denoted by the Greek symbol Γ. You will learn more about this function in later courses on Mathematical Physics:

$$\int_0^\infty \mathrm{d}u\, u^z e^{-u} = \Gamma(z+1).$$

In this problem, we will deal with approximating the factorial using through the Gamma Function, and we will do it using both Gaussian Integration and the Taylor Series we discussed earlier.

(a) Start off by looking at the function that's being integrated. Show, by plotting or sketching the functions u^n and e^{-u} what their product looks like, and explain why you'd expect it to have a maximum.

(b) We will approximate the integral of the function you've just plotted as follows: first, we approximate the function as a Gaussian with some mean and standard deviation, and then we'll then use the formula for Gaussian integration to compute the integral. Start by writing the integral as:

$$n! = \int_0^\infty \mathrm{d}u\, e^{-(u-n\log u)}.$$

The function in the integral decays exponentially fast, so the terms in the integral that will contribute the most will be when the function $u - n\log u$ is minimum. Show that the function $f(u) = u - n\log u$ has a *minimum* when $u = n$.

(c) Do a Taylor Series expansion of the function $f(u) = u - n\log u$ about this minimum to get:

$$f(u) = (n - n\log n) + \frac{(u-n)^2}{2n} + \dots$$

(d) Use this to show that you can approximate

$$n! \approx n^n e^{-n} \int_0^\infty \mathrm{d}u\, \exp\left(-\frac{(u-n)^2}{2n}\right)\left\{1 + \dots\right\}.$$

(e) Now, the integral above *nearly* looks like a Gaussian integral, except for the limits. However, it turns out that if we change the integral's lower limit from 0 to $-\infty$, the error introduced is tiny.

Optional Exercise: Integrate the following expressions numerically to find the functions below and plot them on the same graph for some range of (integers) n:

$$I_1(n) = n! \qquad I_2(n) = n^n e^{-n}\int_0^\infty \exp\left(-\frac{(u-n)^2}{2n}\right) \qquad I_3(n) = n^n e^{-n}\int_{-\infty}^\infty \exp\left(-\frac{(u-n)^2}{2n}\right).$$

Next, plot the relative error associated with the functions $I_2(n)$ and $I_3(n)$, i.e.

$$\frac{\Delta I_2}{I_1} = \frac{I_1(n) - I_2(n)}{I_1(n)} \qquad \frac{\Delta I_3}{I_1} = \frac{I_1(n) - I_3(n)}{I_1(n)}.$$

Beyond which value of n does their difference become less than 1%?

(f) Show that we can now approximate[1]

$$n! \approx n^n e^{-n}\int_{-\infty}^\infty \exp\left(-\frac{(u-n)^2}{2n}\right) = n^n e^{-n}\sqrt{2\pi n}\,\left\{1 + \dots\right\}$$

[1] In physics, it is often sufficient to ignore the $\sqrt{2\pi n}$ term, so that the Stirling Approximation is often simply written as: $n! \approx n^n e^{-n}$. Here is a much quicker proof of that identity, where all we do is replace a sum by an integral:

$$\log(n!) = \sum_{i=1}^n \log i \approx \int_1^n \log x \mathrm{d}x = n\log n - n \qquad \Longrightarrow \qquad n! = n^n e^{-n}.$$

However, the method described in the exercise is better, as it provides a systematic way to find higher order correction terms.