DS 15: Vector Calculus

Philip Cherian

April 30, 2020

1 Types of integrals

I'm going to enumerate the different types of integrals that you will often see in physics; I'm surprised that this is not often done in mathematical physics courses, despite creating quite a lot of confusion for students. We will often be dealing with multiple integrals over space: integrals along paths, over areas, and over volumes. However, there are some subtleties.

1.1 Line Integrals

Open line integrals: Very often in physics we need to compute the integral of a quantity along a curve. A good example of this is to compute the amount of work done when we move an object in a force field. Thus, the work done in moving from a point a to a point b in force field is given by

$$W = \int_a^b \mathbf{F} \cdot \mathbf{dl}.$$

Such an integral is said to be an "open" line integral, as the path is "open".

Closed line integrals: One could imagine a path which closes in on itself. We could then ask ourselves what the value of the quantity is after moving around such a closed path. For example, in school you might have learnt about Ampere's circuital law, which states that the line-integral of the magnetic field around a closed loop is simply related to the current enclosed:

$$\oint_C \mathbf{B} \cdot \mathbf{dl} = \mu_0 I_{\text{enc}}.$$

Such an integral over a closed loop is said to be a "closed" line integral.

Think about this for a minute and convince yourself that an open line integral is bounded by two points, which a closed line integral has no such "boundary".

1.2 Surface Integrals

We could also think of integrating a vector field over a surface, as was discussed in the lecture. For example, if you're interested in the "flux" of water that goes through a given cross-section of a pipe, you are interested in such a surface integral. In this example, the orientation of the surface would certainly matter: a surface perpendicular to the flow would certainly have a much larger flux than one that is parallel to the flow.

Open surface integrals: Surfaces, just like lines, can either be open or closed. An open surface is like a sheet of paper or a sail; a closed surface is like a balloon. Imagine these in your head, and then convince yourself that an **open surface is bounded by a closed curve**. An example of an open surface integral is the amount of (say) water flowing through an arbitrary 2-dimensional surface in a pipe at a given instant of time.

$$\text{Flux} = \iint_S \mathbf{v} \cdot \mathrm{d}\mathbf{A}.$$

Closed surface integrals A closed surface has no bounding curve. An example of a closed surface integral is the **net** amount of water flowing out of an enclosed volume at a given instant of time.

$$\text{Net Flux} = \oiint_S \mathbf{v} \cdot \mathrm{d}\mathbf{A}.$$

The reason that I have emphasised the word "net" is because you could have water flowing through a pipe, for example, but you could have the net flow of water out of a surface to be zero, since the amounts that flow in and out could cancel each other.

1.3 Volume Integrals

The last type of integral we have in a volume integral. Suppose you're interested in the total amount of mass or charge that is within a region. This would be the integral of the density over that volume, something like:

$$M = \iiint_V \rho(x, y, z)\,\mathrm{d}V.$$

An "open" volume is obviously bounded by a closed surface. It is important to realise that because we live in 3-dimensional space, **there is no such thing as a closed volume integral**.[1]

2 The Gradient

We have already seen the gradient in some detail in this course. It turns out that many physical quantities can be written in terms of **gradients** of scalar quantities called **potentials**. This is very important, since from school you should be sufficiently traumatised by vectors to know that it's very easy to make mistakes when adding vectors.

(a) Convince yourself that since a conservative force is defined by its work being independent of the path taken to move between two points a and b, that the work

$$W = \int_{a\to b} \mathbf{F} \cdot \mathrm{d}\mathbf{l} = \varphi(b) - \varphi(a).$$

(b) Show that this is equivalent to saying that $\mathbf{F} = -\nabla\varphi$.

[1] Try to imagine it: going by induction, a closed volume would not have a surface as a boundary. Can you imagine something like that in three dimensions?

3 The Divergence and the Curl

These two "derivatives" are derivatives of vector fields. You could informally think of a vector field is as a collection of little floating "arrows" attached to points in space. For example, a vector field might represent the velocity of the air in a room: at each point in space, you can ask the question "How fast and in what direction is the wind moving at this point?", and represent that with a vector that is "pinned" (so to speak) to that point in space. The room is then "filled" with little arrows, one at every possible location.

It turns out (as mentioned in the lecture) that for most common vector fields, the divergence and curl of a vector field are all that are required to completely specify the field.[2]

The divergence of a vector field is, very crudely, a measure of how much more "vector" is flowing out of an infinitesimal volume element than is flowing into that infinitesimal volume element. The curl of a vector field is, ever more crudely, a measure of the **microscopic** rotation of an infinitesimal volume element due to the vector field.[3]

(a) Calculate the Divergence and Curl of the following vector fields:

  (i) $\mathbf{v} = \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$,

  (ii) $\mathbf{v} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$,

  (iii) $\mathbf{v} = \nabla\left(|r|^2\right)$ ($\mathbf{v}$ is the *gradient* of the scalar r^2.)

(b) Is the following force field conservative?

$$\mathbf{F} = \left(4y^2 + \frac{3x^2y}{z^2}\right)\hat{\mathbf{x}} + \left(8xy + \frac{x^3}{z^2}\right)\hat{\mathbf{y}} + \left(11 - \frac{2x^3y}{z^3}\right)\hat{\mathbf{z}}$$

3.1 Theorems in Vector Calculus

There are some very important theorems in Vector Calculus that every physicist should be able to write out even if they are woken up in the middle of the night. The most important ones are The Divergence (or Gauss's) Theorem, and Stokes' Theorem. We won't prove them here, but what they basically say is that for any vector field:

The Divergence Theorem: The (closed) surface integral of (the normal component of) an arbitrary vector is related to volume integral its divergence over the volume *interior* to the surface. This sounds complicated, but it's written quite simply as:

$$\oiint_{\substack{\text{closed}\\\text{surface}}} \mathbf{v}\cdot d\mathbf{A} = \iiint_{\substack{\text{enclosed}\\\text{volume}}} \nabla\cdot\mathbf{v}\, dV.$$

Stokes' Theorem: The (closed) line integral of the (normal component of) an arbitrary vector over a loop is related to the (open) surface integral of its curl over the volume. Again, this is simply written as:

$$\oint_{\substack{\text{closed}\\\text{loop}}} \mathbf{v}\cdot d\mathbf{l} = \iint_{\substack{\text{enclosed}\\\text{surface}}} (\nabla\times\mathbf{v})\cdot d\mathbf{A}.$$

[2] This is something called the Fundamental Theorem of Vector Calculus or Helmholtz's Theorem. It holds, I believe, for all vector fields that fall off faster that $1/r^2$ as $r \to \infty$. Don't worry if you don't understand this, I'm not sure I do, either.

[3] This particular description is frankly more wrong than right. Take a look here for a slightly better (but longer) explanation.

4 The Continuity Equation

This equation is so important that it deserves a section of its own. It can be found hidden in a variety of different equations in your physics curriculum (I guarantee that you will see it at least once every year in your physics courses at Ashoka). The reason for this is that continuity equations are a stronger, local form of conservation laws: they appear whenever we have a quantity that is being conserved *locally*. Let's try to derive this equation in a very simple way:

(a) Imagine that you have an infinitely long pipe, in which water is flowing. No one knows where this water comes from, or where it's going. You're interested in a small section of the pipe. The flow of the water is not necessarily steady: some places have it moving faster than others. Imagine a small volume of this pipe and ask yourself how the total amount of water in it $Q(t)$ is related to the net "current" of water $I(t)$ that flows out of the cross-sectional area that surrounds it. Show that:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -I.$$

(b) This is a "global" conservation law: it relates the total amount of water at a given instant of time in a region to the total current flowing out of the region's boundary at that time. However, there is no constraint on the size of the volume. What if we made this an infinitesimally small volume? Convince yourself that by definition:

(i) The total amount of water in any volume V is given by $Q(t) = \iiint_V \rho(x, y, z, t)\,\mathrm{d}V$, and

(ii) The total current of water flowing out of a surface S is given by $I(t) = \iint_S \mathbf{j}(x, y, z, t) \cdot \mathrm{d}\mathbf{A}$.

(c) Use the divergence theorem to show that if $Q(t)$ is a *conserved* quantity, then

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

5 Vector Identities

Here are some very important identities that you will use repeatedly, especially in your course in electrostatics.

(a) Show that the divergence of the curl of a vector field is **always** zero. i.e. that for any vector field **F**:

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

(b) Show that the gradient of any scalar field ϕ has no curl. i.e. that for any scalar field ϕ,

$$\nabla \times \left(\nabla \phi\right) = 0.$$

5.1 Some applications of these identities

(a) We've discussed that a **conservative force** is one that can always be written as the gradient of some scalar 'potential'. Show that this is equivalent to saying that a conservative force always has **zero curl**.

(b) In your course in Electromagnetism, you will see that a static (or time-independent) electric field **E** is found to have no curl. Show that this means that we can write the electric field in terms of a **scalar potential** ϕ.

(c) In your course on Electromagnetism, you will see that the magnetic field always has no divergence.[4] Show that this means that you can write the magnetic field in terms of a **vector potential A**.

(d) **Slightly non-trivial:** You will learn next year that the entirety of Electromagnetism can be described by "four" equations, known as Maxwell's Equations:[5]

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{(Coulomb's Law)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t} \qquad \text{(Faraday's Law and Lenz's Law)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(No magnetic monopoles)}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \qquad \text{(Ampere's Circuital Law with Maxwell's Displacement Current)}$$

These equations have contained within them a **lot** of hidden physics which is not immediately apparent. For example, they imply that **charge must be conserved**. As we have seen, conservation of any quantity Q means that the density of that ρ and its current density **j** must **always** satisfy the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

Use the above vector identities as well as Maxwell's Equations to show that charge is conserved. **Hint:** You actually only need two of the four equations.

[4]This is related to a very cool property of the magnetism, namely that there are no magnetic monopoles in the universe. In other words, magnetic charges (unlike electric charges) always appear in pairs. It is impossible to just have a "north-pole magnet" that doesn't have a south-pole.

[5]You also need a fifth equation known as the Lorentz Force law which describes how the electric and magnetic fields translate into forces, but we'll ignore that for now.