DS 8 :
The Legendre Polynomials

Mathematical Physics 2

Spring 2019

1 The Legendre Equation

The Legendre Equation

$$\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+p(p+1)y=0, \tag{1.1}$$

where p is a constant, is of great importance in physics. As we shall see later, this differential equation appears in a range of problems in classical as well as quantum systems which possess spherical symmetry. We will attempt to find a series solution about the origin.

1. Is the origin (a) an ordinary point, (b) a regular singular point, (c) an irregular singular point?

   Justify your answer.

2. From your answer to the previous question, choose an appropriate series solution $y(x)$.
3. Substituting this solution into Equation (1.1), find a recurrence relation between the coefficients of the series.
4. Use these coefficients to show that $y(x)$ can be written as a sum of two infinite series, one with only even powers of x, and one with only odd powers of x:

   $$y(x) = a_0\, y_{\text{even}}(x) + a_1\, y_{\text{odd}}(x)$$

5. Calculate the radius of convergence of these series, for general p.
6. Now, supposing p is a positive integer, show that one of the series truncates. What happens to the other series? (Consider the cases when p is both even and odd separately.)
7. **Bonus:** Looking at Equation (1.1), would there be a way to find out an *upper bound* on the radius of convergence of the series solutions?
8. **Bonus:** Classify any singular points this equation may have.

2 THE GENERATING FUNCTION

You have already encountered these polynomials (henceforth referred to as $\{P_n(x)\}$) in the multipole expansion in Electromagnetism. Knowing that

$$\frac{1}{\sqrt{t^2 - 2xt + 1}} = P_0(x) + P_1(x)t + P_2(x)t^2 + \ldots + P_n(x)t^n + \ldots$$

1. Show that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$
2. $P_{2n+1}(0) = 0$ and $P_{2n}(0) = (-1)^n \frac{1\times 3 \ldots \times(2n-1)}{2^n n!}$

3 ORTHOGONALITY OF POLYNOMIALS

The set of Legendre Polynomials is orthogonal for $x \in [-1,1]$. You will now show that

$$\int_{-1}^{+1} P_n(x)P_m(x)\mathrm{d}x = \begin{cases} 0 & \text{if} m \neq n \\ 2/(2n+1) & \text{if} m = n \end{cases} \tag{3.1}$$

1. For $m \neq n$, begin by showing that you can write Equation (1.1) as

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)P'_m(x)\right) + m(m+1)P_m(x) = 0.$$

2. Write out a similar equation for $P_n(x)$. Multiply the first by $P_n(x)$ and the second by $P_m(x)$ and subtract them.
3. Show that you can simplify the resulting equation into:

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)(P_m P'_n - P_n P'_m)\right) + \left(n(n+1) - m(m+1)\right)P_m P_n = 0$$

4. Integrating this equation from -1 to $+1$, show that

   $$\int_{-1}^{+1} P_n(x)P_m(x)\mathrm{d}x = 0, \qquad m \neq n$$

5. For $m = n$, begin by using the generating function to write

   $$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

6. Do the same for $P_m(x)$ and multiply the two series. Use the result from the case $m \neq n$ to simplify the resulting series.

7. Integrating from -1 to $+1$, show that you can write

$$\sum_{n=0}^{\infty} \left(\int_{-1}^{+1} P_n^2(x)\mathrm{d}x \right) t^{2n} = \frac{1}{t} \ln\left(\frac{1+t}{1-t} \right)$$

8. Express the RHS as a series and compare coefficients to find $\int_{-1}^{+1} P_n^2(x)\mathrm{d}x$.