DS 9 :
Applications of the Legendre Polynomials

Mathematical Physics 2

Spring 2019

1 Properties of Legendre Polynomials

We saw in the last class that the Legendre Equation is given by:

$$\left(1-x^2\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+p(p+1)y=0, \tag{1.1}$$

1. Using the above equation – or any properties of the Legendre Polynomials that you already know, show that for all $P_n(x)$,

   $$\int_{-1}^{+1} P_n(x)\mathrm{d}x = 0.$$

   Hint: One way (there are others) is to begin by rewriting Equation (1.1) in a form more suitable to this integral.

2. You know that the Generating Function $G(x,t)$ of the Legendre Polynomials is given by:

   $$G(x,t)=\frac{1}{\sqrt{1-2xt+t^2}}.$$

   a) Differentiating with respect to t, show that:

   $$(1-2xt+t^2)\frac{\partial G}{\partial t}=(x-t)G.$$

   b) Using the definition of the Generating Function in terms of the Legendre Polynomials, show that:

   $$(n+1)P_{n+1}-2xnP_n+(n-1)P_{n-1}=xP_n-P_{n-1}$$

c) Show that this reduces to

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$$

d) Now differentiating G with respect to x, show that:

$$P'_{n+1}(x) - 2xP'_n + P'_{n-1} = P_n, \quad n \geq 1.$$

e) From this, now show that:

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$$

3. Use the above results to expand the following functions in the interval $[-1, 1]$:

   a) $f(x) = \begin{cases} 0, & -1 \leq x \leq 0 \\ 1, & 0 \leq x \leq 1 \end{cases},$

   b) $f(x) = |x|$

2 Some Applications of Special Functions

1. Consider the vibration of a rectangular membrane, fixed on a frame. Write down the equation that the vibrations satisfy.

2. Write down the general solution $u(x, y, t) = F(x, y)T(t)$. Show that the system of partial differential equations reduces to:

$$\frac{1}{v^2}\frac{\ddot{G}}{G} = \frac{1}{F}\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)$$

3. Argue that the only way this is true is if both sides must be equal to **a constant**. What should the sign of this constant be? (This is not trivial, think about it carefully, the rest of the argument depends on it.) Then show that the above equation reduces to the system of equations: $dotG + \lambda^2 G = 0$ and $\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) + v^2 F = 0$

   Derive the relation between v and λ.

4. To solve for $F(x, y) = H(x)Q(y)$ use the same method to derive two ODEs for $H(x)$ and $Q(y)$.