Projects :
Nonlinear Dynamics

Mathematical Physics 2

Spring 2019

Due: April 28, 2019

LIMIT CYCLES

Consider the following system of equations:

$$\dot{x} = \omega_1 y + x\left(\mu - x^2 - y^2\right)$$
$$\dot{y} = -\omega_2 x + y\left(\mu - x^2 - y^2\right)$$

where ω_1, ω_2, and μ are parameters that can be **positive or negative**.

LOCAL STABILITY ANALYSIS

1. Find the fixed points (x^*, y^*) of this system.
2. Linearise these equations and write the Jacobian $J(x, y)$.
3. Perform a local stability analysis about the points (x^*, y^*).

GLOBAL STABILITY ANALYSIS FOR $\omega_2 = \omega = \omega_1$

1. A global analysis is usually very difficult explicitly. However, $\omega_2 = \omega_1 = \omega$, is a rare exception. Write out the above equations in terms of ω, and show that there exists a co-ordinate transformation such that the coupled equations are decoupled. i.e. show that there exists some $u(x, y)$, $v(x, y)$ such that $\dot{u} = f(u)$ and $\dot{v} = g(v)$. As a result, solve these equations analytically.
2. Show that the number of fixed points depends on μ, and classify them in every case.
3. Show that for some choice of μ, almost all trajectories spiral into a circle of radius $\sqrt{\mu}$, which is therefore a limit cycle.
4. Plot graphs in which you can vary μ and see the resulting trajectories in the $x - y$ plane.

EXPLORATION

1. Choose ω_1 and ω_2 to be distinct. Set ω_1 to some value (say 0.5). Then,
   a) Fix μ to a non-zero value and describe how the phase portrait changes as you vary ω_2.
   b) Keep ω_1 and ω_2 fixed at some interesting value and vary μ and see how the behaviour of the system changes.
2. Classify all the fixed points that you see in the system, mention the nature of the fixed points and compute the index to confirm your result.
3. Find out for what range of values (ω_1, ω_2, and μ) you will get:
   a) Limit cycles.
   b) Saddle points.

   by varying parameters in your plot

Self-Organisation and Evolution

Consider the following model: a group of $n \geq 2$ molecules that are capable of replicating on their own, but can also catalyse each other's replication. Thus, each element in the ecosystem favours the formation of others. A simple model of such a system could be the following dimensionless equation:

$$\dot{y}_i = y_i \left(y_{i-1} - \sum_{j=1}^{n} y_j y_{j-1} \right), \quad i = 1, 2, 3, \ldots, n$$

The variables y_i represents the concentrations of the molecule i, and therefore $y_i > 0$ for all i. Furthermore, $y_0 = y_n$.

Case of $n = 2$

1. Write out the above equations for $n = 2$, and find and classify all fixed points given $y_i > 0$.
2. Let us define two new variables $z_1 = y_1 + y_2$, and $z_2 = y_1 - y_2$. Show that $z_1(t) \to 1$ and $z_2(t) \to 0$ as $t \to \infty$.
3. Use this to conclude that, as $t \to \infty$, $(y_1(t), y_2(t)) \to \left(\frac{1}{2}, \frac{1}{2}\right)$.
4. Plot the phase portrait on the computer, and explain anything interesting that you see.
5. Compute the index for all the fixed points in the system.

Case of $n = 3$

1. Now study the more complicated case of three such interacting elements. Find out as much as you can.

Close Encounters of the Periodic Kind

Suppose we are being harvested by alien beings from another planet. A simple model of this would be to imagine that our population in Delhi would grow logistically, and that alien influence would lead to some form of harvesting.

Constant Harvesting

1. Write out the Logistic Equation, and then include a term with constant harvesting H. Write it in a dimensionless form, in terms of some x, τ, and h that are suitably defined.

2. Show using a plot how the vector field looks for different values of the parameter h.

3. Show that there exists a 'critical' $h = h_c$, where the solutions change their form. Show this both from your graph as well as analytically.

Periodic Harvesting

1. We now move on to another case, where the harvesting varies periodically in time.[1] It might seem likely that – since the harvesting is periodic – the population should also vary periodically over time. Begin with a simple example of purely sinusoidal harvesting. Repeat your analysis for the earlier part, using the equation:

   $$\dot{x} = rx(1-x) - h(1 + a\sin t),$$

   assuming $r, h > 0$, and $0 < a < 1$. Plot the solutions to this equation, and vary the parameters.

2. Show from your plot that – in spite of the humans being harvested periodically (with a period of 2π) – if $h > r/4$ there are solutions in which our population does not vary periodically. What happens to the population at large times in this case?

3. Show that if

   $$h < \frac{r}{4(1+a)},$$

   there exist periodic solutions of period 2π. Further show that for some values of x, you get a stable limit cycle, while for others you get an unstable limit cycle. Identify these values.

   Try to interpret your results.

4. What happens for

   $$\frac{r}{4(1+a)} < h < \frac{r}{4}?$$

5. **Bonus:** Redo the analysis with harvesting frequency as a parameter [i.e. $\sin(\omega t)$ instead of $\sin(t)$].[2]

[1] Perhaps the aliens are not too fond of Delhi summers, and so decide to go bother someone else until a more convenable time of the year.

[2] Too much pollution during the winters - they don't want to be around then either

FORCED OSCILLATORS

Consider the following forced non-linear oscillator:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x + b x^3 = F_0 \sin \omega t$$

THE UNFORCED, UNDAMPED OSCILLATOR

1. Take $\gamma = 0 = F_0$. In this case, solve the above equation exactly in terms of Elliptic Functions, and plot the solution for $x(0) = 1$ and $\dot{x}(0) = 0$.

THE FULL DAMPED DRIVEN OSCILLATOR

1. Now consider the full damped driven non-linear oscillator. Use Mathematica to plot the phase portrait and x as a function of t for a range of parameters F_0, γ (Take $\omega_0 = 1$ and $\omega = 2$). Can you find two distinct steady states with quite different amplitudes as you tune parameters? What can you say about their frequencies?
2. Calculate the index for any fixed points that you find.
3. **Bonus:** Take a Fourier Transform of your data and see what – if anything – you can determine about these states.
4. Compare the above case with another driven damped oscillator, this time with quadratic damping:

$$\ddot{x} + \eta \dot{x}|\dot{x}| + x = F_0 \cos 2t$$

   Use Mathematica to plot the phase portrait and x as a function of t for a range of parameters F_0, η. Do you get limit cycles here? If so, how many distinct steady state behaviours can you get by tuning the parameters – compare and contrast with the previous case.