DS1:
Relativity and the Lorentz Transformations

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We will be expanding on what Dr. Phookun did in class yesterday by deriving the Lorentz transformations in a simple and then in a slightly difficult (though considerably more elegant) way.

1 A Recap

1. Let us begin with a short recap: retrace in your notes the steps that were followed in class to obtain the following transformations:

$$\begin{aligned} \Delta z' &= a_{11}\left(\Delta z - \frac{v}{c}c\Delta t\right) \\ c\Delta t' &= a_{11}\left(c\Delta t - \frac{v}{c}\Delta z\right) \end{aligned} \tag{1.1}$$

   Argue on physical grounds that a_{11} must

   a) Be only a function of v, and

   b) Depend only on **even** powers of v.

   Hint: You may heuristically obtain the inverse transformations and spend some time looking at them.

2. Write down a generic form of $a_{11}(v)$ as a 'power' series with some arbitrary coefficients.

   Hint: A generic power series of some function $f(x)$ can be written as:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

3. Write the pair of equations (1.1) in matrix form, i.e.

$$\begin{pmatrix} \Delta z' \\ c\Delta t' \end{pmatrix} = \Lambda \begin{pmatrix} \Delta z \\ c\Delta t \end{pmatrix} \tag{1.2}$$

   where Λ is the **transformation matrix**.

2 Finding a_{11}: A Simple Approach

1. Argue that the inverse transformation is given by

$$\begin{pmatrix} \Delta z \\ c\Delta t \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} \Delta z' \\ c\Delta t' \end{pmatrix} \tag{2.1}$$

2. Explicitly calculate Λ^{-1} in matrix form.
3. Comparing Equations (1.2) and (2.1), find the form of a_{11} using a physical argument.
4. **Bonus:** What is the determinant of Λ? Can you think of what this value implies in general for transformations? Think of some examples of physical transformations where the determinant could be different. What would that mean?

3 Finding a_{11}: A More Elegant Approach

Disclaimer: *On some inspection, it was found that this question is wrong. However, it is wrong for a very interesting reason. If someone can coherently explain to me what is wrong with it, they will probably get a bonus mark and – which is much more important – will rightly be able to say with confidence that they truly understand Special Relativity.*

In this method we will attempt to reconstruct the entire Lorentz transformations from **infinitesimal** transformations.

The way to do this is to realise that you can 'construct' the entire transformation through a *succession* of infinitesimal transformations. In other words

$$\texttt{result} = \lim_{n\to\infty}\left(\texttt{identity} + \frac{\texttt{action}}{n}\right)^n$$

You should have already seen this in your first Mathematical Physics course. We essentially break up a finite `action` into a large number of infinitesimal parts and act them each successively, taking the large n limit. Formally, the right hand side is just the definition of the exponential, but as an **operator**.

In our case, the `result` is just the transformation matrix Λ. The method we are going to follow is simple: we will look at an infinitesimal change in Λ, find its exponential (remember that it's an operator!), and this will give us the information about the whole transformation.

1. Show that you can write the matrix Λ as

$$\Lambda = I_2 + \omega + \mathcal{O}\left(\frac{v^2}{c^2}\right) \tag{3.1}$$

   where I_2 is the 2×2 identity matrix, ω is a 2×2 matrix, and $\mathcal{O}(\cdot)$ means we ignore terms of order $(\cdot)$ and higher.

2. Convince yourself that

$$\Lambda = e^{\omega}$$

   What assumption have you had to make to do this?

3. The function of a matrix is generally defined by a series. In our case, the function is just the exponential, and so we can write

$$e^{\omega} = \sum_{n=0}^{\infty} \frac{\omega^n}{n!}$$

   In general, finding the powers of a matrix is something you would not wish on your worst enemy. However, it turns out that in this case the individual powers can be obtained by just multiplying the first two or three. So, calculate

   a) ω^2

   b) ω^3

   c) ω^n

   Hint: It turns out that the odd powers are all proportional to each other, and the even powers are all proportional to each other.

4. Using the general formula for ω^n, calculate the matrix e^{ω}.

   Hint: You will get two sets of terms, those containing the odd powers and those containing the even powers. Can you arrive at functions that these terms describe? You may use the fact that

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\dots$$
$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\dots$$

5. **Bonus:** Find the determinant of this matrix. Does it agree with the earlier determinant you calculated? Should it?