DS2 & DS3:
Relativity and the Lorentz Transformations

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1 Matrix Multiplication

One of you (most likely one who does not like to multiply matrices) will be asked to come to the board and do this. Work it out, it's not hard.

1. Write down the rotation matrix in 2+1 dimensions (i.e., for the coordinates t, x, y).
2. Write down the Lorentz boost in terms of the parameter ϕ (defined in Dr. Phookun's homework assignment) for the same coordinates.
3. Show that these two operations do not **commute**. Physically, this means that it matters which one you act before the other. i.e. the two situations
    a) `boost` ← `rotation` ← `vector`
    b) `rotation` ← `boost` ← `vector`

   do not give the same answer.
4. What does this mean for the matrices?

2 Calculating partial derivatives

1. Write down the Lorentz transformations and the inverse Lorentz transformations.
2. Using the theorem from calculus that says that

$$\mathrm{d}f(u, v) = \frac{\partial f}{\partial u}\mathrm{d}u + \frac{\partial f}{\partial v}\mathrm{d}v$$

show how $\frac{\partial f}{\partial y}$ transforms, if y is an arbitrary variable.

3. Setting y as each of the variables x', y', z', and t', calculate the transformation laws for $\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}, \frac{\partial}{\partial t'}$.
4. Now do the same thing for x, y, z, and t.

3 Detecting atmospheric muons

Muons are fat, short-lived cousins of electrons. Being highly unstable, they decay into an electron and two neutrinos very quickly, in around $2\mu s$. Suppose at some time $t = 0$ there are N_0 muons, then at some time t, there will be

$$N = N_0 e^{-t/\tau} \tag{3.1}$$

Muons aren't created naturally anywhere on the Earth as most energies do not come close to what is required to create them. They are, however, created around $10km$ above the surface of the Earth when cosmic rays collide with particles in the upper atmosphere. They are usually very energetic, with speeds close to c, say $v = 0.95c$.

1. Imagine for a minute that the Special Theory of Relativity is not true. If a million (10^6) muons are created in the upper atmosphere, how many would you expect to detect on the surface of the Earth given the above parameters?
2. It turns out that out of every million muons created, around $50,000$ were detected on the surface of the Earth. Early on, this created a dilemma for physicists, but as you will see, this is easily resolved if we take Special Relativity into account.

   Place yourself in the rest frame of the laboratory (say S). You may call the frame S' that in which the muon is at rest. What has changed by introducing Special Relativity? As a result, how many muons would you expect to detect now?
3. If it's true that around $50,000$ muons are detected on the surface of the Earth for every million created, would their average speed be greater than or less than the $0.95c$ that you used in this problem?
4. **Bonus:** Do the same calculations from the muon's frame of reference.
5. Does the above effect occur due to length contraction or time dilation?

4 Not for the faint-hearted: Reconstructing Transformations from the Infinitesimals

Consider a transformation that **preserves only the angles** and orientations between any two vectors. For simplicity, we'll work only in 2 dimensions.

1. What is the determinant of this transformation matrix?

   a) 1

   b) 0

   c) Any positive number

   d) Any real number

   Justify your answer.

2. We would like to obtain the infinitesimal transformation matrix with minimal effort. Let us further assume that this transformation treats both x and y axes symmetrically. Argue that you can write the infinitesimal matrix (C_ϵ)as

$$C_\epsilon = (1+\lambda)\begin{pmatrix} 1 & a_{12} \\ a_{21} & 1 \end{pmatrix} \tag{4.1}$$

   What do you think λ represents?

3. Since these transformations only preserve angles of vectors, argue that if you had two vectors X_1 and X_2,

$$X_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad \text{and} \qquad X_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

   then,

$$X_1' \cdot X_2' = \text{constant} \times (X_1 \cdot X_2) \tag{4.2}$$

   where $A \cdot B = A_i B^i$, and $i = 1, 2$.

4. Since we don't know the complete transformations, we will have to work with the infinitesimal transformations to try and find the relationship between a_{12} and a_{21}.

   Using Equation (4.2), show that $a_{12} = -a_{21}$.

   Hint: Remember, you are dealing with **infinitesimals**, they behave slightly differently from normal numbers: all terms involving any power of an infinitesimal other than 1 can be ignored.

5. Thus, we have

$$C_\epsilon = (1+\lambda)\begin{pmatrix} 1 & b \\ -b & 1 \end{pmatrix} \tag{4.3}$$

   Find the general transformation matrix C in this case.

6. **Bonus:** What can you say about this matrix?