DS10: Potentials and the Method of Images

Solution set

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29th October, 2018

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Table of contents

The Multipole Expansion

Question 1

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$$\begin{aligned} \mathbf{E}_{\text{dipole}} &= -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}\right) \\ &= \frac{2p\cos\theta}{r^3}\hat{r} + \frac{p\sin\theta}{r^3}\hat{\theta} \end{aligned} \tag{2}$$

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Now use the fact that $\mathbf{p} = (\mathbf{p}\cdot\hat{r})\hat{r} + (\mathbf{p}\cdot\hat{\theta})\hat{\theta} = p\cos\theta\hat{r} - p\sin\theta\hat{\theta}$ and the answer should follow. (See Griffiths' Electric Field of a Dipole – pg 153 in my edition, and Problem 3.33).

Charge Distributions

Question 1

We will be using the differential form of Gauss's law: $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0$. Thus, the charge distribution is given by

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The electric field is usually assumed to go to zero far away from the charges. Or, if this isn't given to us, we sometimes use symmetry to guess the form of the field (for example, in the case of infinite sheets, etc.).

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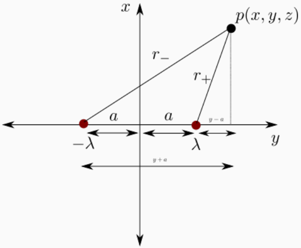
The electric field is usually assumed to go to zero far away from the charges. Or, if this isn't given to us, we sometimes use symmetry to guess the form of the field (for example, in the case of infinite sheets, etc.). However, in this case, the distribution is infinite with no symmetries we can exploit. In fact, it is a badly posed problem: there can be no answer.

(Try calculating the field using Gauss's law, and you'll get an integral that diverges.)

Equipotential Surfaces

Question 1



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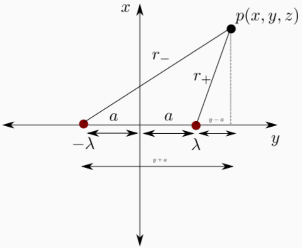
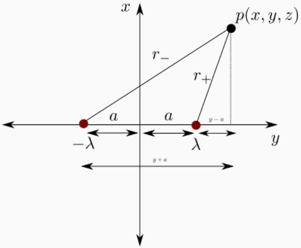


You should already know the answer for the potential for such a '2D dipole' configuration:

$$V = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_+}{a}\right) - \frac{-\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_-}{a}\right) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_-}{r_+}\right) \qquad (5)$$



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Thus, we have that:

$$V(x, y, z) = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{(y+a)^2 + x^2}{(y-a)^2 + x^2}\right) \tag{6}$$

Question 2

Let us set the potential to be constant (V_0). Then, taking the exponential on both sides in the previous equation,

$$\left(\frac{(y+a)^2+x^2}{(y-a)^2+x^2}\right) = e^{\frac{4\pi\epsilon_0 V_0}{\lambda}} = C \tag{7}$$

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$$x^2+y^2+a^2-2ay\left(\frac{C+1}{C-1}\right)=0$$

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$$\begin{aligned} x^2 + y^2 + a^2 - 2ay \left(\frac{C+1}{C-1} \right) &= 0 \\ x^2 + (y - y_0)^2 = R^2 \implies x^2 + y^2 + y_0^2 - R^2 - 2yy_0 &= 0 \end{aligned} \tag{8}$$

Comparing, we see that these are circles centered at y_0, with radius R.

Question 2 (contd.)

$$y_0 = a\left(\frac{C+1}{C-1}\right)$$
$$R = 2a\frac{\sqrt{C}}{|C-1|} \tag{9}$$

Exercise: Show that in terms of V_0 this is

$$y_0 = a\coth\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)$$
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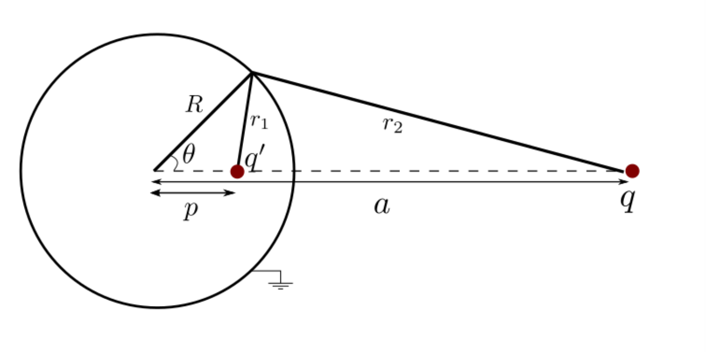
1. When $\lambda \to \infty$ Well... not really a problem.
2. When $V_0 \to 0$ Both the centre and radius of the circle go to ∞! But wait, that's just the $x-$axis, our reference! $\implies$ No problem.

The Method of Images

Questions 1,2, and 3



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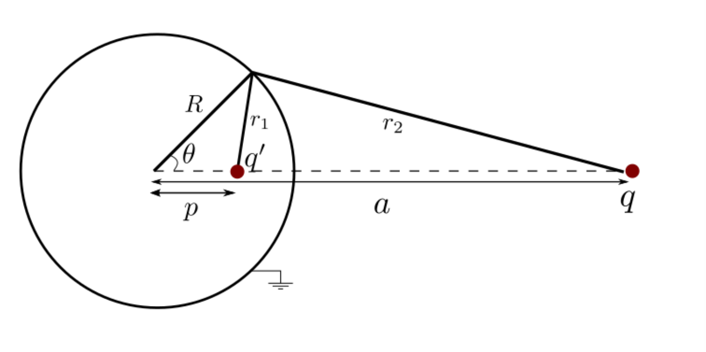

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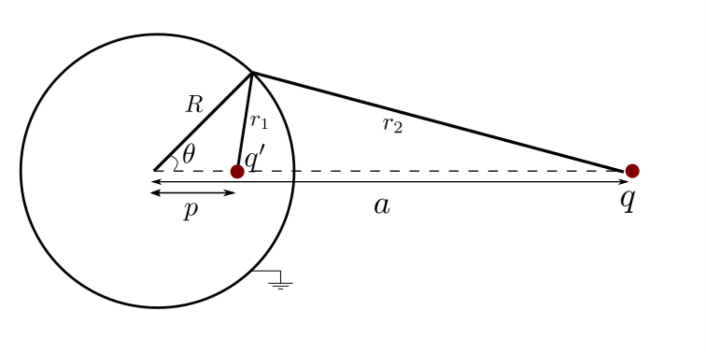

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3. If the potential is zero on the surface, then

$$\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_2}+\frac{q'}{r_1}\right)=0 \implies \frac{q'}{r_1}=-\frac{q}{r_2} \tag{11}$$

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4. The induced charge on the must be the same as the **image** charge.
5. But the induced charge on the shell is negative if q is positive. Thus, so is the image charge.

Question 5

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Thus, we get two equations:

$$\begin{aligned} q^2 &= q'^2\frac{a}{p} && \implies q' = -q\sqrt{\frac{p}{a}} \\ \frac{p}{a}\left(R^2+a^2\right) &= \left(R^2+p^2\right) && \implies p = \frac{R^2+a^2}{2a} \pm \frac{1}{2}\sqrt{\frac{(R^2+a^2)^2}{a^2} - 4\frac{R^2a^2}{a^2}} \end{aligned} \qquad (14)$$

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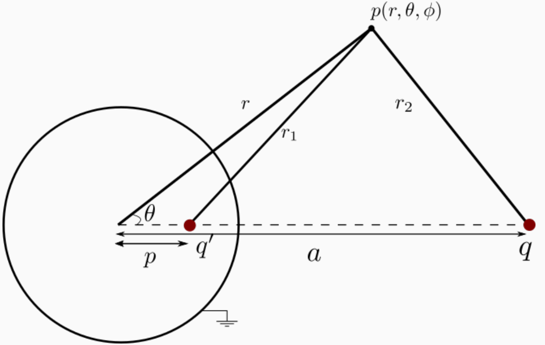
Solution:

$$\begin{aligned} q' &= -q\frac{R}{a} \\ p &= \frac{R^2}{a} \quad \text{or} \quad p = a \end{aligned} \qquad (15)$$

Question 6



1. Consider an arbitrary point outside the shell $p(r,\theta,\phi)$. To find the potential at p, we use the same technique as before, except we replace $R \to r$.

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$$V(p) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_2} + \frac{q'}{r_1}\right) = \left(\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} + \frac{-q\frac{R}{a}}{\sqrt{r^2 + p^2 - 2rp\cos\theta}}\right)$$
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2. Thus, same problem, except with two image charges:

$$q' = -q\frac{R}{a}$$
$$q'' = 4\pi\epsilon_0 R V_0 \tag{17}$$
$$V_{\text{ungrounded}}(p) = V_{\text{grounded}}(p) + \frac{q''}{4\pi\epsilon_0 r}$$

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