DS10:
Potentials and the Method of Images

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1 The Multipole Expansion

We stopped the multipole expansion of two charges (q and $-q$) separated by a distance d at the dipole term. Calculate the Electric Field of this charge distribution. Show that it can be written as

$$\mathbf{E}_{\text{dipole}} = \frac{\left(3(\vec{p}\cdot\hat{r}) - \vec{p}\right)}{4\pi\epsilon_0 r^3}$$

2 Charge distributions

1. Suppose the Electric Field in a region has the following form:

$$\mathbf{E}(\mathbf{r}) = \frac{A\hat{r} + B\sin\theta\cos\phi\hat{\phi}}{r} \tag{2.1}$$

   with A and B as constants. Compute the charge density.

2. Suppose an Electric Field has the following form:

$$\mathbf{E} = ax\hat{x} \tag{2.2}$$

   with a being a constant. What is the charge distribution and density? Now calculate the charge distribution and density for a different problem, when:

$$\mathbf{E} = ay\hat{y} \tag{2.3}$$

Show that it has the same charge density. Explain how the same charge density can produce two **different** electric fields.

3 Equipotential Surfaces

Consider two infinitely long charged wires with charge densities λ_+ and λ_- running parallel to the z axis (as you have already solved before).

1. Find the potential at any point (x, y, z) due to this configuration using the z axis as your reference point.
2. Show that the equipotential surfaces are **cylinders** of circular cross-section. Calculate their axes, and center, for a given potential (say V_0).

HINT: The equation of a circle of radius R centred at (x_0, y_0) is $(x-x_0)^2+(y-y_0)^2 = R^2$.

4 The Method of Images for a Spherical Conductor

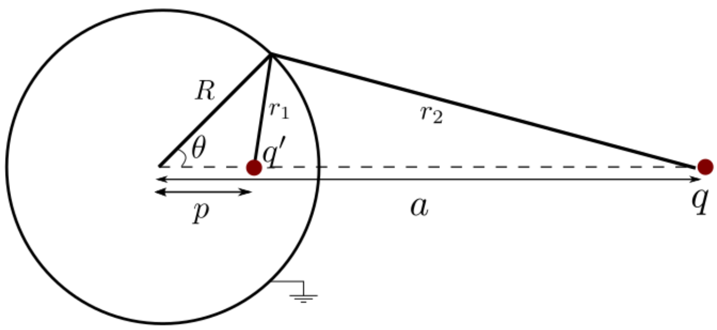

Figure 4.1: A spherical conductor

Consider a grounded spherical conducting shell of radius R, and a charge q at a distance $a > R$ from its centre, as shown in Figure (4.1).

1. Convince yourself that space has been divided into two distinct regions. What are they?
2. What does it mean for the sphere to be grounded?

3. We will now attempt to solve this problem by creating an analogous problem with an image charge of some magnitude q', at some distance p from the centre of the sphere. We will need to determine both of these parameters. Use the fact that the sphere is grounded to establish a relationship between q', q, r_1, and r_2, so that the sphere is an equipotential surface.

4. Different points on the shell will be parametrised by the angle θ. Thus, the above relation should hold for all θ, even though r_1 and r_2 will change for different points on the shell. Thus, we will now try to rewrite the above equation in terms of the constant distances of the problem (R, a, p) and θ. Show then that the above equation reduces to:

$$\frac{q'}{(R^2+p^2-2Rp\cos\theta)^{1/2}} = \frac{-q}{(R^2+a^2-2Ra\cos\theta)^{1/2}}$$

5. We will now try to determine the two free parameters we have in the problem (q' and p). Before we begin, convince yourself that q' and q must have the opposite sign. Now, squaring both sides, collect the terms together intelligently and find the values for p and q'.

   HINT: The $\cos\theta$ term which appears on both sides of the equation is arbitrary (it goes from -1 to 1 depending on the point), but the equation must be satisfied irrespective of its value. Thus the coefficients of this term should be independently equal.

6. How many of values of p and q' do you get? What are they? Calculate the potential at a point exterior to the conductor.

7. Solve the same problem but now for a sphere that is fixed at some constant voltage $V = V_0$. Calculate the potential at a point exterior to the conductor.

8. Argue that the distribution of charges on the surface of the conductor in the previous two cases is a function of θ only. Then calculate it for both the previous cases.

   HINT: You may use the fact that the electric field (and hence the potential) at the surface of a conductor is related to the local charge density by:

$$\mathbf{E} = \sigma/\epsilon_0 \hat{\mathbf{n}}\Big|_{\text{at surface}}$$

   Convince yourself that

$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$$

Begin by showing that in our case,

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R}$$

9. Imagine a dipole placed outside the sphere. Calculate the positions and magnitudes of the image charge(s). Calculate the potential at a point exterior to the conductor. (You will need to write the earlier parts in *vector* form).