DS11:
Curiosities

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1 Field Lines

We will describe a method to calculate **orthogonal trajectories** to certain surfaces. Since you know that the field lines are always locally perpendicular to the equipotential surface, this general technique could – in principle – allow us to calculate the field lines for a certain charge distribution. Consider the following two-dimensional system: a cross-section of of two concentric elliptical cylinders, between which we wish to find the field lines.

1. Convince yourselves that the **family** of equipotential surfaces can be written as:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = c$$

2. Write out the differential equation that these curves satisfy. This is a a point by point relation between, say, $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the point (x, y) locally.[1] Find $f(x, y)$, if

$$y' = f(x, y)$$

3. You should now attempt to find the differential equation satisfied by the **field lines**, i.e., the trajectories orthogonal to the surface at every point. We keep one of the parameters the same.[2] A given curve passing through a point (x_0, y_0) has a slope $f(x_0, y_0)$. The **orthogonal** trajectory through the same point (x_0, y_0) should have a slope $-1/f(x_0, y_0)$! Thus, find the differential equation satisfied by the orthogonal trajectories.

4. Show that these trajectories are **parabolas**.

[1] Note that c cannot appear in this equation, as it is a parameter which characterises its solutions.
[2] i.e. the one with respect to which the derivative was taken – in our case, x.

2 Trajectories in Magnetic Monopole fields

Looking at Maxwell's equations, you should be slightly disappointed, because they don't seem perfectly symmetric between the Electric and Magnetic fields. Wouldn't it be nice(™) if the equations were more symmetric? Notice that in order to do this, we would require a non-zero divergence of B, i.e. a **magnetic** charge density, ρ_m.

1. Write down the magnetic field of a **point** magnetic charge. You may use one of Maxwell's equations, or your ingenuity.[3]
2. Write down the equation of motion of an **electrically** charged particle of charge e in this field.
3. Perform a dot product of this equation with
   a) $\mathbf{v}$,
   b) $\mathbf{r}$

   to get two different equations. **Hint:** You may use that

   $$\mathbf{r}\cdot\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \frac{1}{2}\frac{\mathrm{d}^2 r^2}{\mathrm{d}t^2} - v^2$$

   but work it out at home to convince yourselves.
4. Show that you can write $r = \sqrt{v^2t^2 + b^2}$. What are the "initial" conditions that lead to this solution? Interpret b physically.
5. Use the conservation of the **magnitude** of angular momentum to calculate $\theta(t)$.

3 Magnetic Monopoles

Let's now see if such monopoles can in fact exist.

1. Let us now try to find the vector potential $\mathbf{A}$ that produces this field. Does the following vector potential work? If yes, why? If no, why not?

   $$\mathbf{A}^{(I)} = \frac{q_m}{r}\frac{1-\cos\theta}{\sin\theta}\hat{\phi} = \frac{q_m}{r}\tan\left(\frac{\theta}{2}\right)\hat{\phi} \tag{3.1}$$

   What about

   $$\mathbf{A}^{(II)} = -\frac{q_m}{r}\frac{1+\cos\theta}{\sin\theta}\hat{\phi} = \frac{q_m}{r}\cot\left(\frac{\theta}{2}\right)\hat{\phi} \tag{3.2}$$
2. Calculate the divergence and curl of $\mathbf{B}$ (this is tricky, so be very careful).

[3] Indeed, both are advised, though not necessarily in that particular order.