Assignment 1: The Taylor Series and Harmonic Motion

Due: January 28, 2022 (Friday) **Marks: 15**

1 Planetary Orbits and Simple Harmonic Motion

This problem will help you understand the orbit of a planet around the Sun, without having to know much mechanics to do it. You should have already learnt in your course on *Classical Mechanics* that the effective potential energy of a planet in orbit around the Sun is given by the following expression when the planet is much lighter than the Sun:

$$V_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{GMm}{r},$$

where L is the angular momentum, a conserved quantity, M and m are the masses of the Sun and planet respectively, and r is the radial distance of the planet from the Sun, which varies as the planet goes around, since the orbit is elliptical.

(a) Show that V_{eff} has a minimum, and find the point $r = r_0$, where it occurs. Do a dimensional check on the expression you get for r_0. Does the fact that V_{eff} has a minimum suggest stability or instability? Stability with respect to what? **[2]**

(b) Argue that the minimum at $r = r_0$ represents a a *circular orbit*. Calculate this frequency and time period of this circular orbit, which we will call ω_o and τ_o respectively. **[2]**

(c) Going back to V_{eff}, define a dimensionless radial distance by dividing r by r_0. Show that, in these units, the form of V_{eff} suddenly becomes very neat:

$$V_{\text{eff}} = A\left(\frac{1}{2\rho^2} - \frac{1}{\rho}\right),$$

where $\rho = r/r_0$ and A is some combination of the different parameters of the problem that you have to find. **[2]**

(d) Since we are interested in small deviations about the circular orbit, let us now define a new variable ϵ that represents the deviation from the $\rho = 1$ (i.e., some deviation from $r = r_0$). Show (trivially) that:

$$V_{\text{eff}} = A\left(\frac{1}{2}\frac{1}{(1-\epsilon)^2} - \frac{1}{(1-\epsilon)}\right),$$

if we define $\epsilon = 1 - \rho$ to be the deviation from the mean position $\rho = 1$. **[2]**

(e) Next, by a method of you choosing, show that you can expand the two terms in the sum so that

$$V_{\text{eff}} = A\left(-\frac{1}{2} + \frac{1}{2}\epsilon^2 + \mathcal{O}(\epsilon^3)\right) \quad \Longleftrightarrow \quad V_{\text{eff}} = A\left(-\frac{1}{2} + \frac{1}{2r_0^2}(r - r_0)^2\right),$$

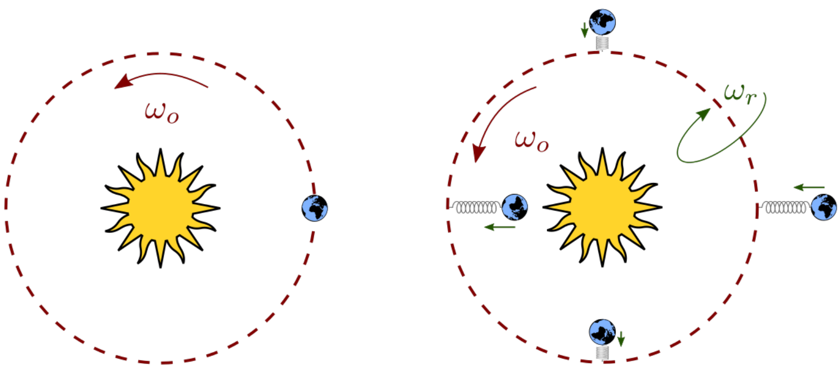
if $\epsilon \ll 1$, and compare this to the potential energy of a simple harmonic oscillator. Find the "spring constant" of the associated oscillator. **[4]**

(f) Find the frequency of this "radial" oscillation, and call it ω_r. Use this to find τ_r, the time period of radial oscillations. How are τ_o and τ_r related? **[1]**

(g) Can you use the relation between τ_o and τ_r to show that small "perturbations" about the circular orbit are *closed* orbits? **[2]**

Hint: In order to do this, perhaps the picture described in Figure (1) will be helpful: you can imagine the planet initially minding its own business and travelling around in a circular orbit, denoted by the dashed line. Now, imagine a malevolent alien entity comes by and – in a moment of spite – gives the planet a little "kick" in the radial direction (say, away from the sun). You have just shown that since the circular orbit is stable, this will produce a small "harmonic" perturbation, so you can imagine the planet is "attached" to the circular trajectory by a little spring, and that this spring is oscillating with a time period τ_r. Now, try to imagine the trajectory that this combined system executes. (It demands a little imagination; you could alternatively also try to plot it out on a computer to see what it looks like, but you would need to think carefully about how to describe this system in equations.)

Now, if the orbit is *closed*, it means that the planet should come back to the original position it started from. What should the relation between τ_r and τ_o be for this to happen?



(a) Circular orbit (b) Small Harmonic Perturbation

Figure 1: The perturbation of a circular orbit by a small radial disturbance.

(h) **Bonus:** Repeat the above procedure for an effective potential of the form

$$V_{\text{eff}} = \frac{L^2}{2mr^2} + \alpha r^n,$$

where n is an integer. For which values of n does this potential admit stable orbits? Calculate τ_o and τ_r. Show that $\tau_r = \tau_o/\sqrt{n+2}$. Argue that if $\sqrt{n+2}$ is rational, the orbits are closed. Plot or sketch the orbits for $n = -1$, 2, and 7. **[Bonus 5]**