Assignment 2:
The LCR Circuit

Due: February 4, 2022 (Friday) **Marks: 15**

1 An electrical, driven, damped, oscillator

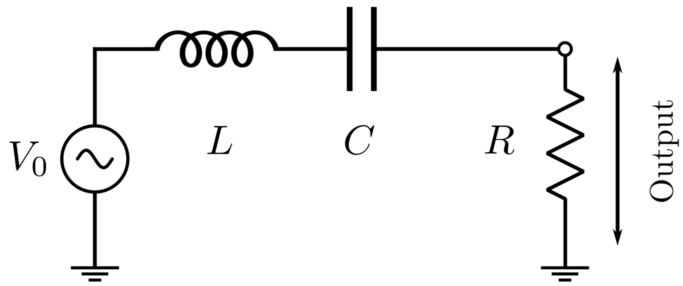

Figure 1: An LCR circuit connected to an AC power supply: the system still "oscillates". However, the resistance introduces a *dissipative* element into the circuit.

We now consider the system shown in Figure (1): unlike the simple LC circuit, the resistance R allows for some energy to flow out of the system (why?), thereby introducing a form of electrical "damping".

(a) Show that this problem can be recast int a damped, driven oscillator in charge Q:

$$\ddot{Q} + \gamma \dot{Q} + \omega_0^2 Q = F_0 \cos(\omega t), \tag{1}$$

where γ, ω_0, and F_0 need to be determined in terms of the constants of the problem. **[2]**

(b) Write out the complete solution to this problem as a sum of the homogeneous and particular solutions. (You may assume that the system is underdamped, i.e. that $\gamma/2 < \omega_0$, so that the solutions are still oscillatory.) **[4]**

(c) What happens to the homogeneous solution as $t \to \infty$? Can you see why the homogeneous solution is often called the *transient* solution in this case? Will the the long-term (steady-state) behaviour of the system be dependent on the initial conditions? **[2]**

(d) Just as in the case of an LC circuit, we have a voltage source which supplies power to the system. But, unlike the LC circuit, we also have a resistor which dissipates this power in the form of heat. Determine expressions for the power input (which depends on the power supply) and the power dissipated by the resistor at steady state. Are these two quantities equal at all times? **[2]**

Hint: In this case, you can use what you remember from school about the power dissipated by a resistor. Remember: the instantaneous power is $P(t) = V(t) \times I(t)$ also varies in time!

(e) Show that the *time-averages* of these two quantities are equal to each other. Explain, physically, why this *must* be the case in the steady state. **[2]**

Hint: The time-averaged power for a process with time-period $T = 2\pi/\omega$ is *defined* to be

$$\langle P \rangle = \frac{1}{T} \int_0^T P(t)\, \mathrm{d}t \tag{2}$$

(f) Show that the time-averaged power input is maximum when $\omega = \omega_0$. This is another way to define resonance. **[1]**

(g) At which values of ω are the steady-state charge and the steady-state current each maximum? Are they both maximum at resonance? What about when $\gamma \to 0$? **[2]**