Assignment 4:
Coupled Oscillators and Symmetry

Due: February 18, 2022 (Friday) **Marks: 15**

1 Dr. Phookun's Coupled Oscillators

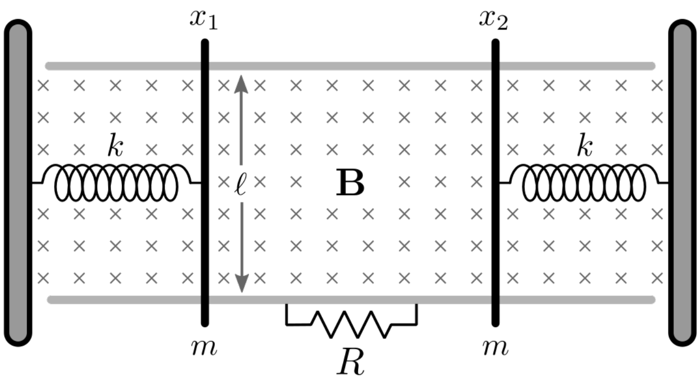

Figure 1: Two identical "free" oscillators are allowed to move on two parallel rails, forming a closed loop with resistance R. A constant external magnetic field **B** is applied throughout the system.

Consider the system shown in Figure (1). Two conducting rods of the same mass m are each attached to a wall with springs of the same spring constant k. These rods are placed on two rails, such that they form the arms of a conducting loop. A single resistance is placed on one of the rails, such that the net resistance in the loop is always R. A magnetic field **B** is established throughout the system, pointing into the paper. As you will show in this exercise, the existence of the magnetic field effectively couples the oscillators in a very interesting way.

(a) You first need to compute the force on the rods due to the presence of the magnetic field. In order to do this, ignore the springs for a moment, and look at one of the rods, keeping the other fixed. Imagine that this rod is moving rightwards with a velocity v_1. This movement changes the area of the loop, which creates an electromotive force $\mathcal{E}$. Find the resulting current I in the loop.[1] **[2]**

(b) The current I produces a magnetic force on the rod. Calculate this force and its direction. **[1]**

(c) Using the answers to the previous part, consider the more general case where both the rods are moving with velocities v_1 and v_2 respectively. Compute the forces on each of the rods, and write out the equations of motion that they both satisfy. **[2]**

(d) You should be able to identify the two normal modes by just inspecting the equations. Write the equations in terms of the two uncoupled coordinates (call them, say, X and x) and write out the solutions. (You don't need to solve them!) **[1]**

[1] You might find it helpful to look at Problem 7.7 in Griffiths' *Introduction to Electrodynamics* (pg. 310 of the 4th Edition).

(e) Describe the long-time behaviour of this system, in terms of the behaviour of the two normal modes. Does one of the normal modes survive longer than the other? If so, explain intuitively why you might expect something like this to happen with this system. **[2]**

(f) Now, suppose that you start off with the initial conditions $x_1(0) = A_1$ and $x_2(0) = A_2$. Compute the initial energy of this system. How much of this energy is lost as $t \to \infty$? If none of it is lost, explain why. If some or all of it is lost, where does it go? (No complicated calculations necessary, you should be able to argue purely on physical grounds what you expect to happen.) **[2]**

2 Periodic Coupled Oscillators and Symmetry

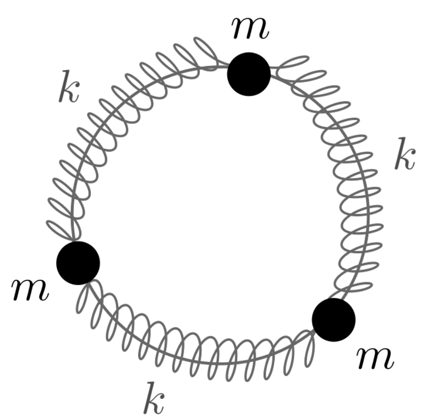

Figure 2: Three identical beads of mass m are placed on a rigid hoop with identical springs of spring constant k attached between them, and are free to move along the hoop.

Note: This question can be solved for the system shown in Figure (2). However, you could alternatively choose to solve it for any number (greater than 4) if you find that more interesting.

(a) Define appropriate displacement variables to describe the system. Describe in words the symmetry that this system possesses, and write out the symmetry matrix $\underline{\underline{S}}$. Use this to find the *form* of the $\underline{\underline{K}}$ matrix. How many terms in this matrix can be set independently? **[1]**

(b) The symmetry matrix is simpler to work with because if you repeatedly act it on a vector you will – after some number of iterations – get back the original vector. Use this idea to find the eigenvalues and eigenvectors of the $\underline{\underline{S}}$ matrix, which you can call β_i and $\underline{A}^i$ respectively. Plot the eigenvalues on the complex plane, and point out which are complex conjugates of each other. **[2]**

(c) Now, explicitly compute the $\underline{\underline{K}}$ matrix, and use it to find the normal mode frequencies ω_i^2 for each normal mode $\underline{A}^i$. You should find that some eigenvectors are complex conjugate pairs: how are their normal mode frequencies related? Use this fact to construct combinations of these pairs that are *real*, and which therefore represent the physical normal modes. **[2]**