Assignment 5: D'Alembert's Solution to the Wave Equation

Due: February 25, 2022 (Friday) **Marks: 15**

1 Transformations of partial derivatives

Let us begin by writing the Wave Equation as partial derivatives between position and time:

$$\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = 0 \tag{1}$$

We are going to attempt to write this equation in terms of two new variables that are a combination of x and t, which we call ζ and η. These quantities are related using

$$\xi = x + vt \qquad \text{and} \qquad \eta = x - vt. \tag{2}$$

We would like to write the wave equation in terms of ξ and η, and you will be able to show that in these variables the equation takes a much simpler form (known as the *canonical* form). In order to do this, we need to make use of the following theorem from calculus that relates a small change in a function (say, g) to its variables. Let us suppose that the function g is a function of two variables u and v. In this case, we can write:

$$\mathrm{d}g = \left(\frac{\partial g}{\partial u}\right)_v \mathrm{d}u + \left(\frac{\partial g}{\partial v}\right)_u \mathrm{d}v \tag{3}$$

(a) Start considering some arbitrary parameter y that both u and v depend on. (For example, u and v could be coordinates, and y could represent time.) Argue that since u and v depend on y, we have:

$$\frac{\mathrm{d}g}{\mathrm{d}y} = \left(\frac{\partial g}{\partial u}\right)_v \frac{\mathrm{d}u}{\mathrm{d}y} + \left(\frac{\partial g}{\partial v}\right)_u \frac{\mathrm{d}v}{\mathrm{d}y} \tag{4}$$

This should be trivial. This relation is just the chain rule for functions of multiple variables. **[1]**

(b) Next, we will imagine that the u and v depend on *two* parameters, y and z, i.e. $u = u(y,z)$ and $v = z(y,z)$. In this case, we need to decide *how* the derivative on the left hand side is taken. For example, we could take it while keeping z constant. Note that we are not performing any new action, we are just redefining what the left-hand side of the above equation *means.* This would mean that:

$$\left(\frac{\partial g}{\partial y}\right)_z = \left(\frac{\partial g}{\partial u}\right)_v \left(\frac{\partial u}{\partial y}\right)_z + \left(\frac{\partial g}{\partial v}\right)_u \left(\frac{\partial v}{\partial y}\right)_z \tag{5}$$

Stare at the above equation for a while, until it sinks in. Now, realise that the above relation must be independent of g, so it must be satisfied by *any* function g that can be expressed by any pair u, v or y, z. Thus, we have the *operator* relation

$$\frac{\partial}{\partial y} = \left(\frac{\partial u}{\partial y}\right)_z \frac{\partial}{\partial u} + \left(\frac{\partial v}{\partial y}\right)_z \frac{\partial}{\partial v} \tag{6}$$

We stop listing explicitly the values kept constant since it should be obvious from the context. Write the corresponding equation for the partial derivative with respect to z, keeping y constant. **[1]**

2 Deriving the d'Alembert solution

(a) Setting $(u, v) \to (\xi, \eta)$ and $(y, z) \to (x, t)$, derive the partial derivatives with respect to x and t as a function of the partial derivatives with respect to ξ and η, using the results you've just shown. **[2]**

(b) Compute the second partial-derivatives with respect to x and t. Note that you just need to compute the successive action of the operator on a "dummy" function g, for example:

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\right) g \equiv \frac{\partial^2 g}{\partial t^2}. \tag{7}$$

Note: In this case, since the derivatives commute with each other, you might feel that using g is unnecessary. However, I guarantee you that if you continue in physics you will encounter cases when the two operators do not commute, and that if you don't use a dummy function, you will end up with wrong relations between operators. I *will* cut marks if you do not use such a function. **[2]**

(c) Now, use the above calculations to show that the wave equation in terms of ξ and η is just **[2]**

$$\frac{\partial}{\partial \xi}\left(\frac{\partial f}{\partial \eta}\right) = \frac{\partial^2 f}{\partial \xi \partial \eta} = 0.$$

(d) Now, simply integrate the above equation twice, once with respect to ξ and once with respect with η. Remember which variable is being kept constant in each case. Use this to show that the general solution $f(\xi, \eta)$ can be written as a sum of two arbitrary functions, $P(\xi)$ and $Q(\eta)$, each of only one variable, i.e. **[2]**

$$f(\xi, \eta) = P(\xi) + Q(\eta) \qquad \Longleftrightarrow \qquad f(x, t) = P(x + vt) + Q(x - vt) \tag{8}$$

3 Plotting the solutions

(a) Choose two arbitrary functions P and Q and some reasonable value of v and using Python plot graphs to show how these solutions move leftwards and rightwards respectively as a function of time, and interact with each other. (If you wish, you may include an animation of these plots.) **[5]**

Note: Try to make your graphs as interesting as possible, but don't add more than two or three images to your assignment. Try to convey as much as you can with the fewest figures. Keep in mind that you need a good (but succinct) figure caption that describes each graph, and a good colour scheme. You can use annotations (like arrows and text) to highlight anything interesting. The more care you put into your graphs, the better your mark will be.