Assignment 7: The Wave Equations in Maxwell's Equations

Due: April 1, 2022 (Friday) **Marks: 15**

1 Working with Electric and Magnetic Fields

(a) Begin by writing out Maxwell's Equations in their most general form (*not* in free space). Now, to obtain a wave equation in (say) **B** just as we did in class, take the curl of the $\nabla \times \mathbf{B}$ equation and show that **[2]**

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{B} = \mu_0 \nabla \times \mathbf{j}. \tag{1}$$

(b) Repeat the same exercise as above, but for the electric field **E**. **[3]**

These equations are *nonhomogeneous* wave equations, since the right-hand sides of these equations work like a "forcing" term. Such equations are quite difficult to solve. However, it is important that you see that (some complicated combination of) ρ and **j** are responsible for "sourcing" electromagnetic waves.

These equations have some more physical content as well: Imagine that you had a source (say, a stationary charge) somewhere in space, and you jiggled it around. Now, imagine that someone far away was measuring the electric field at their (far away) location. Because of the fact that the wave equation involves changes in position as well as time, the information that you have jiggled your charge around does not reach your friend instantaneously: indeed, it turns out (though this is a little complicated to prove) it propagates at a speed c until it gets to her. Thus these equations obey the basic tenet of Special Relativity: information propagates at c.

2 Working with Electromagnetic Potentials

(a) The above wave equations are quite cumbersome to work with, especially since they deal with gradients and curls of the sources. A simpler formalism which gives equivalent results can be obtained from the electromagnetic *potentials*. Begin by using two of Maxwell's Equations to write the electric and magnetic fields in terms of a scalar potential ϕ and a vector potential **A**: **[2]**

$$\begin{aligned}\mathbf{B} &= \nabla \times \mathbf{A},\\ \mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}.\end{aligned} \tag{2}$$

(b) Next, use the other two equations to show that the scalar and vector potentials satisfy the following equations: **[3]**

$$\begin{aligned}-\nabla^2\phi - \frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A}\right) &= \frac{\rho}{\epsilon_0}\\ \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} &= \mu_0\mathbf{j} - \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right)\end{aligned} \tag{3}$$

(c) Now, the above equations don't look very pretty, but there's a way we can simplify them. It turns out that ϕ and $\mathbf{A}$ are not unique: there are many values of ϕ and $\mathbf{A}$ that can give the same $\mathbf{E}$ and $\mathbf{B}$. Show that if you define $\mathbf{A}'$ and ϕ' below in terms of some arbitrary scalar function $\chi(x, y, z, t)$, you would still get the same $\mathbf{E}$ and $\mathbf{B}$. This freedom is called *gauge invariance*. **[1]**

$$\mathbf{A}' = \mathbf{A} + \nabla\chi$$
$$\phi' = \phi - \frac{\partial \chi}{\partial t}$$

(d) Gauge invariance basically means that we can add one constraint to our scalar or vector potentials without changing any physics. Choosing such a constraint is often called "gauge-fixing". There are many famous gauges, but one that is often used is called the **Coulomb gauge**, in which we choose an $\mathbf{A}$ such that $\nabla \cdot \mathbf{A} = 0$.[1]

In this gauge, the equation for ϕ in Equation (3) becomes a very famous equation (called Poisson's Equation) which you should remember from solving problems in Electrostatics. This is a very nice and simple equation to solve, but there is something very strange about finding it in this context. Argue that the equation for ϕ violates the fundamental tenet of relativity. **[2]**

(e) Another useful gauge is called the **Lorenz gauge**, in which we choose

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0. \tag{4}$$

In this gauge, what do the two equations for ϕ and $\mathbf{A}$ become? What role do ρ and $\mathbf{j}$ play in these equations? Argue heuristically that in this gauge Special Relativity is not violated. **[2]**

[1]This is quite easy to do: if you are not convinced, start with an $\mathbf{A}$ such that $\nabla \cdot \mathbf{A} \neq 0$, and try to work out the χ needed that can give you $\mathbf{A}'$ such that $\nabla \cdot \mathbf{A}' = 0$.