Assignment 8: Electromagnetic Waves at an Interface

Due: April 8, 2022 (Friday) **Marks: 15**

1 Transmission and reflection at an interface

We set up the problem just as we did in the Discussion Session – as shown in Figure (1) – with the interface between the two regions being along the xy–plane. In the Discussion Session we saw how the boundary conditions for the electric and magnetic fields at the interface could be written as

$$\left(\;\ldots\; \right) \; e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} + \left(\;\ldots\; \right) \; e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} = \left(\;\ldots\; \right) \; e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}, \qquad \text{at } z = 0, \tag{1}$$

where the quantities in parentheses would be filled in by the actual boundary conditions. We saw how irrespective of what these quantities were, the phases had to all be equal, from which we concluded:

- The xz–plane forms a *plane of incidence*, which contains all the wave-vectors $\mathbf{k}$, as well as $\hat{\mathbf{z}}$,
- The law of reflection: $\theta_I = \theta_R$,
- The law of refraction: $n_1 \sin\theta_I = n_2 \sin\theta_T$.

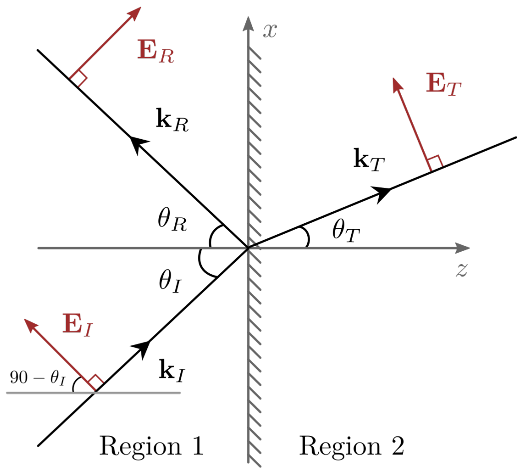

Figure 1: Electromagnetic waves at an interface in the case of parallel polarisation. Since the polarisation must always be perpendicular to $\mathbf{k}$, and we have only one possible direction in which each $\mathbf{E}$ can point.

So far, we have not spoken about anything specific to electromagnetic waves. We will now look at the quantities in the parentheses and show how they effect the electric and magnetic fields at either side of the interface. From your course in Electromagnetism, you should know that at the interface between two media without free charges or currents, the electic and magnetic fields obey:

$$\textbf{(A)}\quad \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \qquad \textbf{(B)}\quad \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}$$
$$\textbf{(C)}\quad B_1^{\perp} = B_2^{\perp} \qquad \textbf{(D)}\quad \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} \tag{2}$$

(a) The boundary conditions above can be a little confusing because they are often written in terms of "parallel" and "perpendicular" components to the *interface*, rather than to the plane of incidence. To avoid confusion, let us write them out in terms of their x, y, and z components. Begin by showing that the above boundary conditions can be written as: **[2]**

$$\begin{aligned}
\epsilon_1 \left(\mathbf{E}_{0I} + \mathbf{E}_{0R}\right)_z &= \epsilon_2 \left(\mathbf{E}_{0T}\right)_z \\
\left(\mathbf{B}_{0I} + \mathbf{B}_{0R}\right)_z &= \left(\mathbf{B}_{0T}\right)_z \\
\left(\mathbf{E}_{0I} + \mathbf{E}_{0R}\right)_{x,y} &= \left(\mathbf{E}_{0T}\right)_{x,y} \\
\frac{1}{\mu_1}\left(\mathbf{B}_{0I} + \mathbf{B}_{0R}\right)_{x,y} &= \frac{1}{\mu_2}\left(\mathbf{B}_{0T}\right)_{x,y}
\end{aligned} \tag{3}$$

(b) We know that electromagnetic waves have a polarisation: i.e., a direction along which their electric and magnetic fields oscillate. This direction can be anything except along the direction of propagation $\mathbf{k}$. We will consider two cases separately. First, let us consider an electromagnetic wave that is oscillating in the plane of incidence, i.e. in the $xz-$plane. This is the case where the polarisation is *parallel* to the plane of incidence. Draw a figure like Figure (1) and compare the components of $\mathbf{E}_I$, $\mathbf{E}_R$, and $\mathbf{E}_T$ (as well as all the $\mathbf{B}$s) to show that **[2]**

$$\begin{aligned}
\epsilon_1 \left(-E_{0I}\sin\theta_I + E_{0R}\sin\theta_R\right) &= -\epsilon_2 E_{0T}\sin\theta_T, \\
\left(E_{0I}\cos\theta_I + E_{0R}\cos\theta_R\right) &= E_{0T}\cos\theta_T,
\end{aligned} \tag{4}$$

(c) Using the laws of reflection and refraction, show that

$$\begin{aligned}
E_{0I} - E_{0R} &= \underbrace{\left(\frac{\mu_1 n_2}{\mu_2 n_1}\right)}_{\beta} E_{0T}, \\
E_{0I} + E_{0R} &= \underbrace{\left(\frac{\cos\theta_T}{\cos\theta_I}\right)}_{\alpha} E_{0T},
\end{aligned} \tag{5}$$

and derive the **Fresnel Equations** for polarisation parallel to the plane of incidence: **[2]**

$$E_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) E_{0I}, \qquad E_{0T} = \left(\frac{2}{\alpha + \beta}\right) E_{0I} \tag{6}$$

(d) Now let us consider polarisation perpendicular to the plane of incidence. In this case, $\mathbf{E}$ would point along the $\hat{\mathbf{y}}$ direction. Write out the boundary conditions (as before) and show that

$$\begin{aligned}
E_{0I} + E_{0R} &= E_{0T}, \\
E_{0I} - E_{0R} &= \left(\alpha\beta\right) E_{0T}.
\end{aligned} \tag{7}$$

Find the Fresnel equations when the wave is polarised perpendicular to the plane of incidence. **[4]**

2 Total Internal Reflection

We will now consider the case where $n_1 > n_2$: our incident wave is moving from an optically dense to an optically rare medium. We have seen in class that Snell's Law indicates that there exists a **critical** angle θ_c such that when $\theta_I = \theta_c$, $\theta_T = \pi/2$. When $\theta_I > \theta_c$, there appears to be no transmitted wave at all.

(a) Remember that the **k** vector for the transmitted wave can be written as (why?):

$$\mathbf{k}_T = |k_T| \left(\sin\theta_T \hat{\mathbf{x}} + \cos\theta_T \hat{\mathbf{z}} \right). \tag{8}$$

Now, use the fact that if $\theta_I > \theta_c$, then $\sin\theta_T > 1$ and $\cos\theta_T$ is imaginary. Use this to show that

$$\mathbf{E}_T = \mathbf{E}_{0T} e^{-\kappa z} e^{i(kx-\omega t)}, \tag{9}$$

where

$$\kappa \equiv \frac{\omega}{c} \sqrt{(n_1 \sin\theta_I)^2 - n_2^2} \qquad \text{and} \qquad k \equiv \frac{\omega n_1}{c} \sin\theta_I. \tag{10}$$

Describe this wave in detail. **[2]**

(b) Using the results from the previous problem, calculate the reflection coefficient for polarisation (i) parallel and (ii) perpendicular to the plane of incidence. **[1]**

(c) All the fields that we used so far were complex numbers. Write out the **real** fields and construct the Poynting vector **S**. Show that *on average*, no energy is transmitted in the $z-$direction. In which direction is the energy transmitted? **[2]**