Assignment 9: Transfer Matrices

Due: April 17, 2022 (Sunday) **Marks: 15**

1 The Transfer Matrix Method

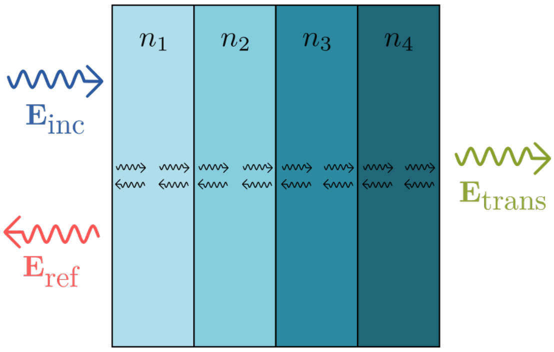


Figure 1: The setup: an electromagnetic wave is incident on a succession of interfaces with different lengths. We are interested in the reflection and transmission coefficients of the entire system.

In the last assignment, we saw what happens when an electromagnetic wave travels across an interface between two regions with different refractive indices. In this assignment, we will consider a more general problem, and introduce a powerful technique of solving it, at least in principle.[1] Consider a problem as set up in Figure (1): an electromagnetic wave is incident from the left on a succession of different media with different refractive indices. As the light reaches the first interface, some of it is reflected and some transmitted. However, we now have a much more complicated problem: the electric (and magnetic) fields in each of the regions is some complicated combination of the reflected and transmitted waves because of multiple reflections and transmissions at every interface.

We will try to work out in complete generality what happens at one interface, and see how we can use this result to generalise transmission across multiple interfaces. In order to begin, we consider the following simplifications: first, we assume only normal incidence. This makes the formulae for the reflection and transmission coefficients quite simple.[2] Next, we will only consider **three** interfaces, with different refractive indices n_1, n_2, and n_3. Remember that the reflection and transmission coefficients across a boundary ij at normal incidence are given by:

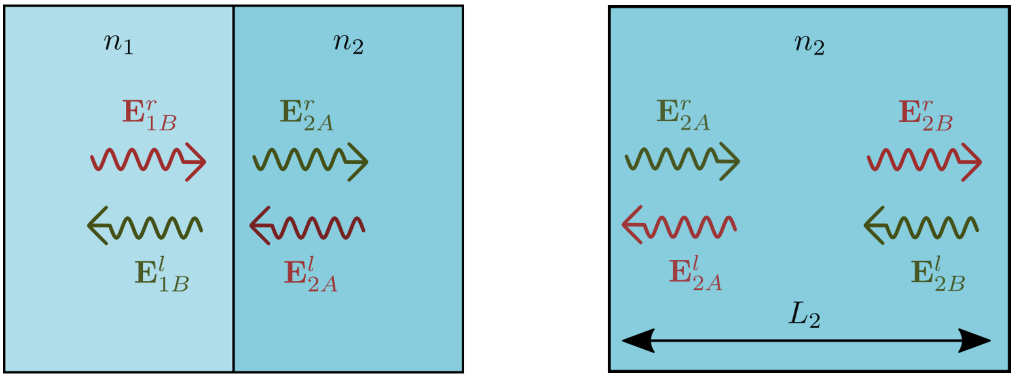
$$r_{ij} = \frac{n_i - n_j}{n_i + n_j} \qquad \text{and} \qquad t_{ij} = \frac{2n_i}{n_j + n_j} \tag{1}$$

Notice how $t_{ij} \neq t_{ji}$, and $r_{ij} \neq r_{ji}$. This is an important result for what we are going to compute.

[1] What "in principle" means in this context is that while you might not be able to solve this problem simply, it is a systematic approach, meaning that you could easily write a numerical method to solve it.

[2] I have not done this for oblique incidence, if anyone feels up to it, I would strongly urge you to at least try to solve that problem using this method.

1.1 The Transfer and Propagation Matrices



(a) Waves at an interface

(b) Waves in a medium

Figure 2: The analysis of our problem breaks down into looking at what happens at an interface, where reflection and transmission occur (left) and what happens in a medium, where the waves simply change phase (right). **Note:** The r does **not** denote reflected, it denotes a *rightward-moving* wave!

Consider an interface somewhere in the middle of our system, as shown in Figure (2a). We will now try to write a matrix that describes exactly what happens when one goes from one interface to the other. The only difficulty in this approach is in the notation, so let us clear that up right now: at every interface, we have a left-moving and a right-moving wave on either side, denoted by the superscripts l and r. Let us denote the waves on the left-hand side with a subscript B (for "before interface") and those on the right hand side with a subscript A (for "after interface") as shown in the figure.

$$E^{l}_{1B} \begin{array}{l} \longrightarrow \\ \longrightarrow \end{array} \begin{array}{l} \text{leftward moving} \\ \text{in region 1 at interface to region 2} \end{array} \tag{2}$$

(a) Argue that **[2]**

$$\begin{aligned} E^r_{2A} &= t_{12}E^r_{1B} + r_{21}E^l_{2A} \\ E^l_{1B} &= r_{12}E^r_{1B} + t_{21}E^l_{2A} \end{aligned} \tag{3}$$

(b) We want to find a relation between the "before" waves and the "after" waves. Rearrange the above equations to find the *transfer* matrix $\underline{\underline{D_{12}}}$ which satisfies

$$\begin{pmatrix} E^r_{1B} \\ E^l_{1B} \end{pmatrix} = \underline{\underline{D_{12}}} \cdot \begin{pmatrix} E^r_{2A} \\ E^l_{2A} \end{pmatrix}, \tag{4}$$

and show – using the form of the transmission and reflection coefficients – that **[4]**

$$\underline{\underline{D_{12}}} = \frac{1}{t_{12}} \begin{pmatrix} 1 & r_{12} \\ r_{12} & 1 \end{pmatrix}. \tag{5}$$

(c) We now have a systematic method to deal with what happens at the interface. But what about what happens *within* a medium? In other words, we need to now relate the waves E^r_{2A} to E^r_{2B}, and so on.

Argue, using the fact that we are working with linear media, that

$$\begin{aligned} E^r_{2B} &= E^r_{2A} e^{+ik_2 L_2}, \\ E^l_{2B} &= E^l_{2A} e^{-ik_j L_2}, \end{aligned} \tag{6}$$

where $k_j = n_j \omega / c$, and L_j is the thickness of the layer. Use this to write the propagation matrix, defined by **[2]**

$$\begin{pmatrix} E^r_{2A} \\ E^l_{2A} \end{pmatrix} = \underline{\underline{P_2}} \cdot \begin{pmatrix} E^r_{2B} \\ E^l_{2B} \end{pmatrix}, \tag{7}$$

1.2 The Transmission Matrix

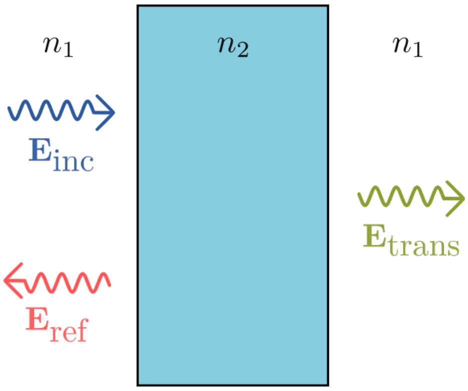


Figure 3: A thin film of refractive index n_2 is placed in a material of refractive index n_1. Thus, our problem has three regions of space, with refractive indices n_1, n_2, and n_1, respectively.

We now have all the elements required to solve a problem with an arbitrary number of layers. However, let us restrict ourselves to the system with three regions described in Figure (3).

(a) Show that any pair of waves in the right-most region can be related to the waves in the left-most region using the transmission matrix M, where **[3]**

$$\begin{pmatrix} E^r_{1B} \\ E^l_{1B} \end{pmatrix} = \underbrace{\underline{\underline{D_{12}}} \cdot \underline{\underline{P_2}} \cdot \underline{\underline{D_{21}}}}_{\underline{\underline{M}}} \cdot \begin{pmatrix} E^r_{3A} \\ E^l_{3A} \end{pmatrix}, \tag{8}$$

(b) What are E^l_{3A} and E^r_{3A}? In other words, what are the leftward and rightward moving waves in the third region? Similarly, what are E^l_{1B} and E^r_{1B}? **[2]**

(c) Show that if we define the coefficients of the matrix $\underline{\underline{M}}$ to be M_{ij}, then the *net* reflection and transmission coefficients are given by **[2]**

$$t = \frac{1}{M_{11}} \qquad \text{and} \qquad r = \frac{M_{21}}{M_{11}}. \tag{9}$$

In the next assignment, you will calculate this exactly and find a very interesting result.