Assignment 11: The Fourier Transform

Due: April 22, 2022 (Friday) **Marks: 15**

1 The Dirac Delta Function

In the Discussion Session, we discussed how the inverse of the Fourier Transform of a function should give us back the same function, which translated into mathematics as the Fourier Integral Theorem:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t' f(t') \int_{-\infty}^{\infty} \mathrm{d}\omega \; e^{i\omega(t'-t)}. \tag{1}$$

In order to show that the above result is true, we need to evaluate the integral over ω. However, this is a strange object, and cannot be evaluated using simple means. In order to evaluate it, we will need to introduce a new object, called the Dirac Delta function $\delta(\tau)$. The function is often described (very loosely) as satisfying the identities

$$\delta(\tau) = \begin{cases} \infty, & \tau = 0, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \mathrm{d}\tau \; \delta(\tau) = 1. \tag{2}$$

However, a far more fundamental definition is the one we are interested in. The Dirac Delta function is *defined* as that function which – given any arbitrary function $g(\tau)$ – satisfies:

$$g(0) = \int_{-\infty}^{\infty} \mathrm{d}\tau \; g(\tau)\delta(\tau). \tag{3}$$

Using this definition, it should be quite clear why we need something like a Dirac Delta in order to prove the Fourier Integral Theorem.

(a) We first attempt to approximate the Dirac Delta in terms of more familiar functions: one way to think of such a function is to imagine it to be a limit of a Gaussian with variance going to 0. Let us consider the function:

$$\delta_\epsilon(\tau) = \mathcal{N} e^{-\tau^2/2\epsilon^2} \tag{4}$$

Compute the value of $\mathcal{N}$ such that **[2]**

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \; \delta_\epsilon(\tau) = 1. \tag{5}$$

(b) Plot $\delta_\epsilon(\tau)$ for three different values of ϵ on the same graph, and show that it seems to satisfy both requirements of the Dirac Delta given in Equation (2) in the limit $\epsilon \to 0$. **[2]**

In what follows, we will assume that

$$\delta(\tau) = \lim_{\epsilon \to 0} \delta_\epsilon(\tau). \tag{6}$$

(c) We will now use this definition of the Dirac Delta to evaluate the integral

$$I(\tau) = \int_{-\infty}^{\infty} \mathrm{d}\omega \; e^{i\omega\tau}. \tag{7}$$

Start by defining a new integral,

$$I_\epsilon(\tau) = \int_{-\infty}^{\infty} \mathrm{d}\omega \; e^{-\epsilon\omega^2} e^{i\omega\tau}, \tag{8}$$

and convince yourselves that

$$I(\tau) = \lim_{\epsilon\to 0} I_\epsilon.$$

Evaluate $I_\epsilon(\tau)$ and show that **[4]**

$$I(\tau) = \lim_{\epsilon\to 0} I_\epsilon(\tau) = 2\pi\delta(\tau) \tag{9}$$

(d) Use the above result to prove the Fourier Integral Theorem given in Equation (1). **[2]**

2 Babinet's Principle

Babinet's principle is an interesting result in Fraunhofer diffraction, which states that

Complementary objects produce the same diffraction pattern, except for the intensity of the central maxima.

Two objects are complementary if one of them is transparent when the other is opaque and opaque when the other is transparent. A single slit is thus complementary to a wire of the same thickness. The aperture functions of both such objects are shown in Figure (1).

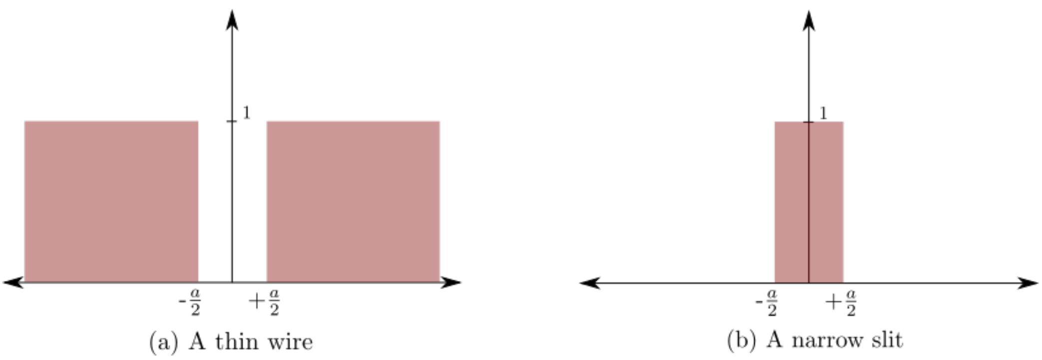

Figure 1: Aperture functions for a thin wire and a single slit, both of width a.

(a) Show mathematically how the aperture functions of an arbitrary aperture and its complement are related. **[2]**

(b) Using the fact that in the regime of Fraunhofer diffraction, the pattern on the screen is related to the Fourier Transform of the aperture function, prove Babinet's Principle for an arbitrary aperture. **[3]**

Hint: You will need to know the Fourier Transform of 1, but you should know that from the previous question.