DS 1: The Taylor Series

Philip Cherian

January 21, 2022

1 Introduction and Derivation of the Taylor Expansion

The Taylor expansion is one of the most important tools in a physicist's toolkit. It allows us to examine the behaviour of different functions *locally*, i.e. in the vicinity of a point, say, where some interesting physics happens. Or the expansion may allow an approximation that makes our calculations easier.

From the definition of the derivative in earlier courses, you should have seen that

$$f(x+\Delta x) \approx f(x) + \Delta x f'(x).$$

We say that such an equation is approximate since it is in general true only up to the first order in Δx. We are interested in finding an expression for $f(x+\Delta x)$ that is true for *all* values of Δx.

To make it clear, we can begin by writing our earlier approximation like this:

$$f(x+\Delta x) = \left(1 + \Delta x \frac{\mathrm{d}}{\mathrm{d}x}\right) f(x).$$

This allows us to interpret the quantity $1 + \Delta x \frac{\mathrm{d}}{\mathrm{d}x}$ as an *infinitesimal translation operator*, since it acts on the function $f(x)$ and *translates* it to the point $x+\Delta x$. By 'infinitesimal' we mean that this is true up to the first order in Δx. However, since such a translation is a *continuous* process, it can be broken into smaller constituent operations. In other words, we could think of a translation of Δx as a succession of n translations of $\Delta x/n$. Now, each translation by $\Delta x/n$ is given by the infinitesimal translation operator

$$1 + \frac{\Delta x}{n}\frac{\mathrm{d}}{\mathrm{d}x},$$

and so

$$\begin{aligned}
f\left(x + \frac{\Delta x}{n}\right) &\approx \left(1 + \frac{\Delta x}{n}\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) \\
f\left(x + 2\frac{\Delta x}{n}\right) &\approx \left(1 + \frac{\Delta x}{n}\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x+\Delta x) = \left(1 + \frac{\Delta x}{n}\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 f(x) \\
&\vdots \\
f(x+\Delta x) &\approx \left(1 + \frac{\Delta x}{n}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x)
\end{aligned}$$

Of course, we could now imagine making n larger and larger, which essentially means that we are dividing the region into finer and finer intervals, which would make this "approximation" closer and closer to the exact value. Thus, we can say that

$$f(x+\Delta x) = \lim_{n\to\infty}\left(1+\frac{\Delta x}{n}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x).$$

In the above expression, you should recognise the definition of the exponential function which can itself be written as a famous series,

$$e^y = \lim_{n\to\infty}\left(1+\frac{y}{n}\right)^n = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots,$$

Our expression for $f(x+\Delta x)$ is thus given by

$$f(x+\Delta x) = e^{\Delta x\frac{\mathrm{d}}{\mathrm{d}x}} f(x),$$

where the first term is an operator, called the *finite translation operator*. (You will hear more about this in your course in Quantum Mechanics at the start of your third year.) When this operator is expressed in terms of the standard series for the exponential, you get the **Taylor Series**:

$$f(x+\Delta x) = \left(\mathbb{1}+\Delta x\frac{\mathrm{d}}{\mathrm{d}x}+\frac{(\Delta x)^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\ldots+\frac{(\Delta x)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}+\ldots\right)f(x).$$

Since there is no longer a requirement that Δx be small, it is conventional to write the Taylor Series about the point $x = a$ as:

$$f(a+x) = f(a) + x\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a} + \frac{x^2}{2!}\left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x=a} + \ldots + \frac{x^n}{n!}\left.\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right|_{x=a} + \ldots. \tag{1}$$

For a Taylor Series to make sense, it must converge. To determine the radius of convergence of a Taylor Series we should think of x as a complex variable. The region of convergence of the series (i.e. the maximum value of Δx for which the expansion gives the right answer) is a disk with the point at which the expansion is being done at the centre, and the nearest singularity of the function on the complex plane at the boundary. (Don't worry if this doesn't make sense to you right now, you can just accept it.) For example, consider the function

$$f(x) = \frac{1}{1-x}.$$

It is a very instructive exercise to expand it about the point $x = 0$, and show that

$$f(0+x) = 1+x+x^2+x^3+\ldots$$

The Taylor Series formula is very useful. However, it has one limitation that it needs an arbitrary number of derivatives to be computed, which is not – in general – simple to do by hand. There is, thankfully, a useful fact about the Taylor Series that can help us: it is **unique**. This means that if you could come up with a series for $f(x)$ by "accident", then it is guaranteed to be the actual series (provided certain mathematical qualifications are satisfied that, frankly, I've not had to use so far so I can safely say can be ignored by the average physics student for a while).

This is useful because the series for

$$\frac{1}{1-x}$$

can be obtained by using the geometric series approach that you might have learnt in school, which is a much easier technique to compute it. Furthermore, if we now wanted to use this to calculate the series for

$$f(x) = \frac{1}{1+x},$$

all we'd need to do is take the series we already have and replace $x \to -x$, so that we have $f(x) = 1 - x + x^2 - x^3 + \ldots$!

2 Problems:

(a) Begin by computing the series for e^x using Equation (1).

(b) Compute the Taylor Series for the following functions using Equation (1):

(i) $\sin(x)$,

(ii) $\cos(x)$,

(c) Show that $e^{ix} = \cos(x) + i\sin(x)$ using the series obtained above.

(d) Derive the series for $\frac{1}{1-x}$ using the geometric series approach. Using only this series (i.e. **without** using Equation (1)), find the series for:

(i) $\dfrac{1}{(1-x)^2}$,

(ii) $\log(1-x)$

(e) **Computation:** Write a program to calculate the partial sums of the first n terms of the Taylor series expansions about $x = 0$ of $\sin(x)$, $\cos(x)$, e^x, and $\log(1-x)$. For each of these functions, plot a graph of the "actual" function as well as two partial sums for different values of n, and explore how well the series converges.