DS 2:
The Driven Harmonic Oscillator

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1 Length and time scales in differential equations

(a) Consider the differential equation for simple harmonic motion.

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -kx.$$

Identify a quantity of dimension time (which we will call T) using the constants of this problem. In other words, find some combination of the constants given in this problem that results in a quantity that has dimensions of time.

(b) Now let's assume that we measure time in units of T. i.e., instead of "some number of seconds have passed", we say "some number of Ts have passed". Show that is just a shift from the variable t to the variable $\eta = t/T$. What are the dimensions of η?

(c) Rewrite the differential equation in terms of η and x. You should be able to see that the problem no longer depends on any parameters that have units. This is known as "nondimensionalising" a differential equation.[1] Can you think of any benefits of doing this?

(d) Now consider, Duffing oscillator, a system that is used to model "stiff springs":

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -kx - \alpha x^3.$$

In this case, we can define *two* quantities of interest: in addition to the earlier time-scale (that we'll still call T), and one with dimensions of length that we'll call ℓ.

(e) What do T and ℓ represent physically? Now let's count time in units of T, and length in units of ℓ. Define two new variables $\eta = t/T$, and $\lambda = x/\ell$.

(f) Rewrite the differential equation in terms of η and λ, just as before. Show that the differential equation can now be written as:

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}\eta^2} = -\left(\lambda + \lambda^3\right).$$

Think about what happens when $x \ll \ell$ and $x \gg \ell$ (i.e., when $\lambda \ll 1$ and $\lambda \gg 1$ respectively). What is this saying "physically"?

[1] See more about this on Wikipedia here: https://en.wikipedia.org/wiki/Nondimensionalization.

2 The driven harmonic oscillator

Consider the system shown in Figure (1), also known as an *LC* circuit.

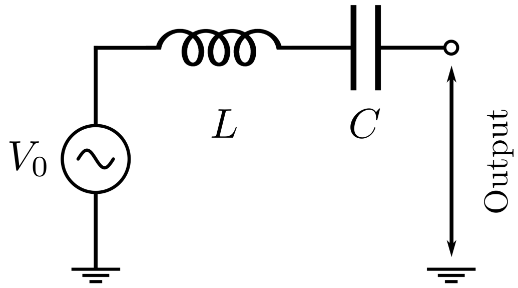

Figure 1: An LC circuit connected to an AC power supply is a very good (and easily constructed) example of an "oscillator" that is being driven by the external power source. However, unless you're being unusually rought with the setup, the system is not oscillating *mechanically*, but rather *electrically*.

(a) Write out the relation that the voltage satisfies in this system. Show that it can be written as as simple harmonic oscillator equation in the *charge* $Q(t)$,

$$\ddot{Q} + \omega_0^2 Q = V(t), \tag{1}$$

where you need to identify the constant "frequency" ω_0.

In what follows, we will consider the (realistic) case of when the power supply is turned on at some time which we take to be $t = 0$. Thus, if ω is the power supply's frequency,

$$V(t) = \begin{cases} 0 & t \leq 0, \\ V_0 \cos(\omega t) & t > 0. \end{cases} \tag{2}$$

(b) What are the initial conditions here? Find the general solution for $t > 0$.

(c) Apply the initial conditions and show that you get

$$Q(t) = \left(\frac{V_0}{L}\right) \times \left(\frac{1}{\omega_0^2 - \omega^2}\right) \times \Big(\cos(\omega t) - \cos(\omega_0 t)\Big).$$

(d) Find the expression for *current* in the circuit, $I(t) = \dot{Q}(t)$.

(e) For which values ω are $Q(t)$ and $I(t)$ maximum? Is this the same value of ω for both the charge and the current?

(f) Write the solution for Q in terms of the quantities:

$$\omega_e \equiv \frac{\omega + \omega_0}{2} \qquad \text{and} \qquad \omega_b \equiv \frac{\omega - \omega_0}{2}.$$

Sketch the resulting waveform when the two frequencies are close to each other.

(g) Now, let ω tend towards ω_0. Take this limit *carefully*, and sketch the resulting function. Does it blow up? If so, when?