DS 6:
Fourier Analysis

Philip Cherian
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1 Introduction

Consider a periodic function $f(x)$ that has some period L. This means that the function repeats itself after every L units. Mathematically, this would mean that

$$f(x+L) = f(x),$$

for every value of x. In other words, if we know what the function looks like over one period, we can sketch it over *any* interval.

Periodic functions arise very often in physical problems, and thus finding a way to represent them in terms of simpler functions would be extremely useful. In 1822, a French mathematical physicist named Jean Baptiste Joseph Fourier was studying the conduction of heat in solids, and found that it should be possible to express periodic functions of some period L in terms of an infinite series of elementary trigonometric functions:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{L}\right). \tag{1}$$

This is known as Fourier's theorem.

Exercise: Show that the above equation implies that $f(x+L) = f(x)$. Note that this is just a *sufficient* condition that a periodic function $f(x)$ be written in this form, not a *necessary* condition, so it is not a proof of Fourier's Theorem.

Fourier's work was quite controversial when it was introduced, mainly due to the fact that the concept of a "function" was not as well defined as it is today. The idea at the time was that $f(x)$ was a function if could be represented as a polynomial, or a finite or infinite combination of elementary functions, like a power series or a trigonometric series. However, if $f(x)$ had gaps or discontinuities, the general thought was that it could not be treated as a legitimate function. However, Fourier's claim was that any arbitrary graph (as we shall see below) could be treated as a legitimate function, expressible in terms of an infinite series of sines and cosines! In fact, during one of Fourier's presentations, Joseph Louis Lagrange – a giant in his own right – apparently exclaimed that Fourier's ideas were "nothing short of impossible."

Shortly after his death, Fourier's achievements were almost forgotten in France. So much so that Victor Hugo in "Les Miserables" could write "*Il y avait à l'Académie des Sciences un Fourier célèbre que la postérité a oublié, et dans je ne sais quel grenier un Fourier obscur dont la postérité se souviendra.*"[1]

[1] "There was a celebrated Fourier at the Academy of Science, whom posterity has forgotten and there is now in some attic an obscure Fourier, whom the future will remember." The "other Fourier" was Charles Fourier, a philosopher, was not related to the physicist.

Thankfully, Hugo's prediction does not seem to have come to pass. Many of the most important mathematical discoveries of the nineteenth century are directly linked to the theory of Fourier series. It took some years before the issues surrounding the definition of a function were completely clarified. It is no accident that the modern definition of a function was first formulated by Dirichlet in 1837, in a paper on the theory of the Fourier Series. The Fourier Series was also central to Riemann's 1854 paper in which he introduced the modern notion of the definite integral.

In what follows, we will describe the applications of the Fourier Series to problems in oscillations and waves, but you will encounter many of the same problems – in almost identical settings – during your *Quantum Mechanics* course next semester.

2 The Fourier Series

There are many equivalent representations of the Fourier Series, all of which differ in their definition of the *period* of the function. In the definition given in the previous section, we had assumed the period of the function to be L. However, it is possible that certain problems would require a period of $2L$, or a period of 2π, and so on. It is straightforward to generalise this to functions of arbitrary period. For now, we will continue to use the definition given above, with the understanding that:

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{L}\right), &&\longrightarrow \quad f(x) \text{ has period } L \\ f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right), &&\longrightarrow \quad f(x) \text{ has period } 2L \\ f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx), &&\longrightarrow \quad f(x) \text{ has period } 2\pi \end{aligned} \tag{2}$$

The strength of the Fourier Series is that the discrete coefficients a_0, a_n, and b_n contain *all* the information about the function $f(x)$. Furthermore, since these coefficients multiply functions that are defined over the entire interval, truncating the above series of trigonometric functions gives us a reasonable approximation of $f(x)$ over the entire period, rather than just near some point like the Taylor Series.[2]

This is also an important result in signal processing: create sinusoids of different periods is relatively simple (remember LC circuits) and combining them together in different proportions can lead to arbitrarily complicated periodic signals on devices like the Digital Oscilloscopes in the laboratory.

We would now like to compute the coefficients a_0, a_n, and b_n, given some function $f(x)$. Before we do this, it's important to establish some important mathematical results.

2.1 Mathematical preliminaries

2.1.1 Odd and even functions

We will eventually want to compute integrals over different intervals, some of them symmetric, and it turns out that the behaviour of a function under reflection about the origin can help us evaluate many of these integrals simply.

[2] Remember that for a Taylor Series, as you go further away from the point of expansion x_0, you need many more terms to approximate the function sufficiently well.

A function is said to be **even** if, under a reflection about $x = 0$, it remains unchanged. In other words, changing $x \to -x$ doesn't change the function. This means that you could replace your $y-$axis with a mirror, and the function and its reflection would look exactly like the entire function would, without the mirror. In contrast, an odd function is one which flips its sign on such a reflection.

$$\begin{aligned} f(x) &= f(-x) && \rightarrow && \text{Even function} \\ f(x) &= -f(-x) && \rightarrow && \text{Odd function} \end{aligned} \tag{3}$$

Odd and even functions behave differently over integrals over symmetric domains (say, over $(-a, a)$).

Exercise: Show that

$$f(x) \text{ is even} \quad \longrightarrow \quad \int_{-a}^{+a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x \tag{4}$$

Exercise: Show that

$$f(x) \text{ is odd} \quad \longrightarrow \quad \int_{-a}^{+a} f(x)\,\mathrm{d}x = 0 \tag{5}$$

A standard example of even and odd functions are the cosine and sine functions respectively. We can try to apply the above results to them.

Exercise: Begin by showing (without doing the integrals!) that

$$\begin{aligned} \int_{-L}^{L} \mathrm{d}x \sin\left(\frac{n\pi x}{L}\right) &= 0 \\ \int_{-L}^{L} \mathrm{d}x \cos\left(\frac{n\pi x}{L}\right) &= 2\int_0^L \mathrm{d}x \cos\left(\frac{n\pi x}{L}\right) \end{aligned} \tag{6}$$

Exercise: Next, evaluate the integrals below to arrive at the answers:

$$\begin{aligned} \int_0^L \mathrm{d}x \sin\left(\frac{n\pi x}{L}\right) &= \frac{(1-(-1)^n)L}{n\pi} \\ \int_0^L \mathrm{d}x \cos\left(\frac{n\pi x}{L}\right) &= 0 \end{aligned} \tag{7}$$

The above results can be consolidated as follows (pay attention to the limits!):

$$\begin{aligned} \int_0^L \mathrm{d}x \sin\left(\frac{n\pi x}{L}\right) &= \frac{(1-(-1)^n)L}{n\pi} & \qquad \int_{-L}^{L} \mathrm{d}x \sin\left(\frac{n\pi x}{L}\right) &= 0 \\ \int_0^L \mathrm{d}x \cos\left(\frac{n\pi x}{L}\right) &= 0 & \qquad \int_{-L}^{L} \mathrm{d}x \cos\left(\frac{n\pi x}{L}\right) &= 0 \end{aligned} \tag{8}$$

2.1.2 Orthogonality

Another important concept that you should already be a little familiar with from linear algebra is the notion of orthogonal functions: functions which are not just linearly independent, but "mutually independent". This is very similar to how vectors in two dimensional space can be considered orthogonal, and we would thus like to generalise the notion of the dot-product of vectors to a vector space of *functions*.

The dot-product in such function vector-spaces, often called a **bilinear form** by the more mathematically inclined, is defined using an integral, for example:

$$\langle f, g \rangle = \int_a^b \mathrm{d}x f(x) g(x).$$

The above definition can be used – for example – to define the norm (the analogue of the "length" of a geometric vector) of a function to be:[3]

$$\int_a^b \mathrm{d}x f(x) g(x) = ||f||_2.$$

The scope of such a definition will be explored in a later course,[4] but right now we will simply restrict ourselves to a condition for orthogonality between two functions f and g over some domain (a, b). Two functions are said to be orthogonal over the domain iff:

$$\int_a^b \mathrm{d}x f(x) g(x) = 0. \tag{9}$$

However, just as in the case of normal (geometric) vectors, we can define the orthogonality of a *set* of vectors. We define a set of orthonormal[5] vectors which are indexed by some i, j, such that:

$$\int_a^b \mathrm{d}x f_i(x) f_j(x) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{10}$$

Happily, the set of all trigonometric sines and cosines forms such an orthogonal set, and can be made orthonormal by appropriate rescaling. You will, over the course of this ear, become familiar with many other orthogonal functions like the Legendre Polynomials, the Hermite Polynomials, the Chebyshev Polynomials and many, many more, all defined over different domains.

2.1.3 Identities for the products of trigonometric functions

To prove that the sines and cosines are orthogonal, we will need to multiply them with each other and perform integrals of the form

$$\int_a^b \mathrm{d}x \sin(k_n x) \sin(k_m x),$$

and such product integrals are not particularly simple to compute using integration by parts. However, there are some trigonometric identities that make our life much simpler.

[3]This is often known as the L^2−norm, as indicated by the subscript 2 in the notation.
[4]You will certainly be introduced to a version of this in your Quantum Mechanics course next semester.
[5]Ortho, meaning orthogonal, and normal meaning that they have been rescaled to have a norm (as defined earlier) of 1.

The three most important identities are the following:

$$\begin{aligned} \sin A \cos B &= \frac{1}{2}\Big(\sin(A-B)+\sin(A+B)\Big) \\ \sin A \sin B &= \frac{1}{2}\Big(\cos(A-B)-\cos(A+B)\Big) \\ \cos A \cos B &= \frac{1}{2}\Big(\cos(A-B)+\cos(A+B)\Big) \end{aligned} \tag{11}$$

The above relations can be proved quite simply using the standard sum identities for trigonometric functions. We will follow a slightly different approach here, using Euler's relation,

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{12}$$

using which we can also define:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \tag{13}$$

Consider the sum of two complex exponentials:

$$\frac{e^{i(A+B)}+e^{i(A-B)}}{2} = e^{iA}\cos B. \tag{14}$$

Taking the real and imaginary parts of the above equation we get

$$\begin{aligned} \frac{\cos(A+B)+\cos(A-B)}{2} &= \cos A \cos B, \\ \frac{\sin(A+B)+\sin(A-B)}{2} &= \sin A \cos B, \end{aligned} \tag{15}$$

Exercise: Now, use the difference of the same exponentials, i.e.

$$\frac{e^{i(A+B)}-e^{i(A-B)}}{2},$$

to show that:

$$\begin{aligned} \frac{\cos(A-B)+\cos(A+B)}{2} &= \sin A \sin B, \\ \frac{\sin(A+B)-\sin(A-B)}{2} &= \cos A \sin B, \end{aligned} \tag{16}$$

2.1.4 Orthogonality of sines and cosines

We are now equipped to derive the main results that make the Fourier Series so useful, the orthogonality of the sine and cosine functions.

Result 1: All elements in the set of cosines $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ are mutually orthogonal in the interval $(-L, L)$, as well as the interval $(0, L)$.

In order to prove this, we need to compute the following integral:

$$\int_{-L}^{L} dx \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \frac{1}{2}\int_{-L}^{L} dx \cos\left(\frac{(n+m)\pi x}{L}\right) + \frac{1}{2}\int_{-L}^{L} dx \cos\left(\frac{(n-m)\pi x}{L}\right) \tag{17}$$

Now, when $n = m = 0$, the overall integral is clearly just $2L$. Furthermore, from our results in Equation (8) regarding the integrals of sines and cosines over the $(-L, L)$ domain, the first integral is clearly zero. If $n \neq m$, then the second integral is zero as well. However, if $n = m$, then the cosine term just becomes 1, and so we have:

$$\int_{-L}^{L} dx \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \begin{cases} 2L & n = m = 0, \\ L & n = m \neq 0, \\ 0 & n \neq m. \end{cases} \tag{18}$$

Exercise: Use the fact that the function being integrated is even to show that

$$\int_{0}^{L} dx \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \begin{cases} L & n = m = 0, \\ L/2 & n = m \neq 0, \\ 0 & n \neq m. \end{cases} \tag{19}$$

Result 2: All elements in the set of sines $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ are mutually orthogonal both in the interval $(-L, L)$, as well as the interval $(0, L)$.

As before, we prove this by using the product formulae we spoke of earlier:

$$\int_{-L}^{L} dx \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \frac{1}{2}\int_{-L}^{L} dx \cos\left(\frac{(n-m)\pi x}{L}\right) - \frac{1}{2}\int_{-L}^{L} dx \cos\left(\frac{(n+m)\pi x}{L}\right) \tag{20}$$

Now, as before, if $n = m = 0$, the integral is just 0. However, if $n = m \neq 0$, then by the same arguments as before, the first integral gives us L, while the second gives us 0. And, as before, if $n \neq m$, both integrals are zero. As a result, we have:

$$\int_{-L}^{L} dx \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \begin{cases} L & n = m, \\ 0 & n \neq m. \end{cases} \tag{21}$$

Exercise: Use the fact that the function being integrated is even (remember, the product of two odd functions is even!) to show that

$$\int_0^L \mathrm{d}x \, \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \begin{cases} L/2 & n = m \neq 0, \\ 0 & n \neq m. \end{cases} \tag{22}$$

Result 3: All elements in the set of sines $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ and the set of cosines $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ are mutually orthogonal both in the interval $(-L, L)$.

Unlike the previous cases, we now have two separate sets. We have already shown that each individual set comprises elements that are mutually orthogonal. As a result, all that is left to do is for us to take one arbitrary element from each set, and to show that their product integral is always zero. We could, of course, expand the function in terms of a sum of sines and cosines, but it turns out not to be necessary. Indeed, since the sine functions are odd and the cosine function is even, their product is an odd function. It should be relatively simple for you to convince yourselves that the integral of an odd function over a symmetric integral is zero, as we have already shown above. Therefore:

$$\int_{-L}^{L} \mathrm{d}x \, \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = 0. \tag{23}$$

Note: Notice how, unlike the previous two cases, the orthogonality only holds over the entire domain $(-L, L)$, and **not** over $(0, L)$.

Exercise: Show that

$$\int_0^L \mathrm{d}x \, \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \frac{\left((-1)^{m+n}-1\right) nL}{\pi\left(m^2-n^2\right)} \tag{24}$$

Set	0 to L	$-L$ to L
$\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$	✓	✓
$\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$	✓	✓
$\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ and $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$	✗	✓

Table 1: Representation of the orthogonality of the different sets of trigonometric functions over different domains.

2.2 Computing the Fourier Coefficients

With all of the mathematical preliminaries completed, we can now focus our attention on computing the coefficients for any arbitrary periodic function $f(x)$. Remember that when our function has a periodicity of $2L$, one could write:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right). \tag{25}$$

Given all of our discussion so far about orthogonality and vector-spaces, this representation should remind you of expressing a geometric vector in terms of some projections along the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions. Just as in that case, when we found the projection by dotting the vector with each of the unit vectors, here too we can compute the analogous projections (i.e. the Fourier Coefficients) using the bilinear form defined above.

We start by realising the since the integrals of sines and cosines over the entire domain $(-L, L)$ is zero, we can write

$$\int_{-L}^{L} f(x)\mathrm{d}x = \frac{a_0}{2}\int_{-L}^{L}\mathrm{d}x + \sum_{n=1}^{L} a_n \underbrace{\int_{-L}^{L}\mathrm{d}x \cos\left(\frac{\pi n x}{L}\right)}_{=0} + \sum_{n=1}^{L} b_n \underbrace{\int_{-L}^{L}\mathrm{d}x \sin\left(\frac{\pi n x}{L}\right)}_{=0}. \tag{26}$$

From where we get our first Fourier Coefficient, and you can see why a_0 is sometimes called the DC component of the series, and represents the average of the function over the domain:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\mathrm{d}x \tag{27}$$

Now, we can obtain the other coefficients by exploiting the orthogonality relations we established before.

Exercise: Show that

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \end{aligned} \tag{28}$$

Notice the limits: if you only had sines or cosines in your series, you could always just integrate from 0 to L. However, the fact that you have both sines *and* cosines means that the integrals *have* to be conducted over the entire period of $2L$, since – as we've established in the previous section – sines and cosines are mutually independent only over the entire period of $2L$.

The best way to get a handle on this is to actually do some calculations of the Fourier Series of some simple functions, which is what we'll be doing in the next section.

3 Applications to the Plucked and Struck String Problems

One of the important applications of Fourier analysis, although perhaps it can be argued that it is more an application of the orthogonality of sines and cosines, is the study of waves on a string.

3.1 Solving the wave equation by separation of variables

We begin by looking for solutions to the wave equation

$$-\frac{1}{v_p^2}\frac{\partial^2 f}{\partial t^2}+\frac{\partial^2 f}{\partial x^2}=0 \tag{29}$$

that can be separated into a product of a function purely of time, and one purely of space. We name these functions $X(x)$ and $T(t)$, and therefore we are looking for solutions of the form:

$$f(x)=X(x)T(t). \tag{30}$$

Plugging this into the wave equation, and keeping in mind that the partial derivative with respect to t sees a function of x as a constant and vice versa, we see that the equation can be written as

$$-\frac{1}{v_p^2}X(x)\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}+T(t)\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}=0, \tag{31}$$

where we replace the partial derivatives with total derivatives because the functions they act on are each functions of only one variable. We can now divide the above equation by $X(x)T(t)$, and separate the terms so that the left-hand side has only functions of t and the right-hand side has only functions of x:

$$\frac{1}{T(t)}\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}=\frac{v_p^2}{X(x)}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}. \tag{32}$$

While the mathematics of this is quite simple, the implications are not as easy to grasp: what this means is that we have two functions, one purely of x and another purely of another *independent* variable t that are **always** equal. In other words, *for separable solutions*, these two quantities must independently be equal to a constant. In other words, we need:

$$\begin{aligned}\frac{1}{T(t)}\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}&=\text{constant},\\ \frac{v_p^2}{X(x)}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}&=\text{constant}.\end{aligned} \tag{33}$$

Are there any restrictions on this constant? Indeed there are! Look at the time equation above: you should recognise that it looks almost exactly like the equation for simple harmonic motion, provided the constant be negative.

Exercise: Show what happens to (say) the time equation when the constant is:

(a) zero,

(b) positive,

(c) negative,

and explain which of the solutions would give *oscillatory* behaviour.

Thus, since we require our solution to be oscillatory in time, we can establish that the constant has to be negative, and so we call it $-\omega^2$. In this case, we can instantly write out the solutions to the equations given above in terms of ω and $k = \omega / v_p$:

$$\begin{aligned} X(x) &= A\cos(kx) + B\sin(kx), \\ T(t) &= C\cos(\omega t) + D\sin(\omega t). \end{aligned} \tag{34}$$

The separable solutions to the wave equation can thus be written as products of sines and cosines! We should be happy about this. Remember why we did all of this: partial differential equation are notoriously hard to solve, and therefore we did not attempt to find the general solution to the wave equation. Instead, we looked for a *class* of solutions that had the time and space components separated. It then turned out that these solutions *happened* to be sines and cosines. Now, from Fourier analysis, we know that *any* periodic function can be represented as a linear sum of sines and cosines. Therefore, it would appear that we can construct *any* general solution using a linear combination of these separable solutions. This is a deep result, spend some time understanding it before we move on to two applications.

3.2 Waves on a string with fixed ends

Let us start off by imposing a constraint on our string: it is fixed firmly at both ends, meaning that the points labelled by $x = 0$ and $x = L$ simply do not move, ever. This is called a **boundary condition**, as it specifies the value of the solution at some boundary, and thereby imposes a condition on the solutions.[6]

The general solution to the wave equation, which we will from now on denote as $u(x, t)$ must thus satisfy the following conditions:

$$u(0, t) = 0, \qquad \text{and} \qquad u(L, t) = 0. \tag{35}$$

Now, we have guessed that the separable solutions can be used to get the general solution, and so it makes sense to see which – if any – of the separable solutions satisfy these conditions.

Given that these are conditions on the position for *all time*, these translate into conditions on the $X(x)$ function, since they must be true at all values of t and must therefore be true *independently* of $T(t)$.

Let us look at each boundary condition at a time:

$$u(0, t) = 0 \quad \Longrightarrow \quad X(0) = 0 \quad \Longrightarrow \quad A = 0. \tag{36}$$

[6] This is somewhat similar to, though quite different in many ways from the initial conditions you are used to seeing in Newtonian mechanics: solving Newton's laws gives you a *family* of solutions, and you pick out only those that start off with some initial position and some initial velocity.

Thus, our solutions cannot have any terms that look like $\cos(kx)$, since these functions are never zero when $x = 0$.

Let us now look at the other boundary:

$$u(L,t) = 0 \quad \Longrightarrow \quad X(L) = 0 \quad \Longrightarrow \quad B\sin(kL) = 0. \tag{37}$$

Now, B cannot be zero, since that would just give us the trivial solution $X(x) = 0$ which would be terribly boring. As a result, only certain values of k must be allowed, so as to ensure that $kL = n\pi$, which in turn would ensure that the solutions always ended up at 0 when $x = L$.[7] As a result, we now see that there are only specific values of $k_n = n\pi/L$ which satisfy the boundary conditions. Remember that the relation between ω and k[8] implies that only certain values of $\omega \equiv \omega_n$ are allowed as well. Therefore, the solution $X(x)$ is given by:

$$X(x) = B\sin\left(\frac{n\pi x}{L}\right). \tag{38}$$

3.2.1 The plucked string

So far, we have imposed no condition on the function $T(t)$. However, we could do that as well. Suppose we had a situation where the string was *plucked*, meaning that it initially starts off with no velocity, and only an initial displacement. The more astute of you will probably see this as another "boundary" condition, except that it is a boundary condition in *time*. In other words, we are stating that at the initial instant of time $t = 0$, the string has no velocity. But how does one define the velocity of a string, you ask? It's simple: remember that we have so far been considering our string to be a set of coupled oscillators whose $y-$displacement denotes the degree of their oscillation, which – in the continuum case – we represent as the function $u(x,t)$. By the same token, it would only make sense to denote the velocity of the string by the rate of change of u with respect to time, for each label x. In other words, the velocity of the string at any given point x is given by

$$v(x) = \frac{\partial u}{\partial t}(x,t).$$

Using this simple definition we see that the condition that the string begin without any velocity means that for every point, at $t = 0$, the string is at rest.

Exercise: Convince yourself that this means that

$$\frac{\partial u}{\partial t}(x,0) = 0. \tag{39}$$

Of course, just as before, we can argue that since this is a condition that only affect the string at one instant of time, and for all possible points x equally, it must impose a condition only on the corresponding "time" function $T(t)$. As a result, we can say that:

$$\frac{\partial u}{\partial t}(x,0) = 0 \quad \Longrightarrow \quad T'(0) = 0 \quad \Longrightarrow \quad B\omega_n = 0. \tag{40}$$

As we have established that $\omega_n \neq 0$, the only possibility is that the string's time-dependence must be purely a cosine. In other words, the condition that the string start off as being *plucked* leads us to say that the

[7] This is very similar to the way that we guessed the solutions for the finite number of normal modes.
[8] The "dispersion" relation $\omega^2 = k^2 v_p^2$

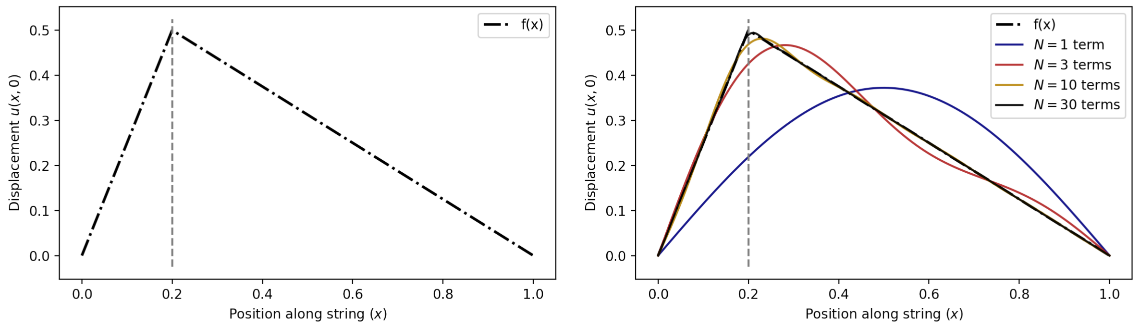
separable solutions must be of the form:

$$f_n(x,t) = A_n \sin(k_n x)\cos(\omega_n t) \qquad k_n = \frac{n\pi}{L}, \quad \omega_n = k_n v_p. \tag{41}$$

Notice how we now introduce a subscript for the coefficient A_n, preempting the fact that each "standing wave" could have its own amplitude. Now, we are almost done with solving the problem: the general solution can be written as a linear sum of all the separable solutions, given that they now satisfy all the boundary conditions (in space and time) described above. And so we must have that:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{v_p n\pi t}{L}\right). \tag{42}$$

All that remains now is to find the amplitudes of each of the separable solutions (henceforth called "modes"), which we can find from the initial conditions (i.e., *how* the string was plucked at $t = 0$.)



(a) Displacement profile of the string **(b)** Successive approximations using Fourier expansion

Figure 1: A string plucked at some distance $a = L/5$, with $L = 1$. The displacement profile on either side of $x = a$ is linear (left). This function can be approximated by a finite number of terms from the Fourier expansion (right). The approximation improves as N increases, but even $N = 10$ is a faithful approximation.

Exercise: Consider a string plucked using the profile described above, with and $a = L/5$, and $h = 1$. Show that the initial displacement profile of the string can be written as:

$$u(x,0) = \begin{cases} \left(\frac{h}{a}\right)x & x \le a, \\ \left(\frac{h}{L-a}\right)(L-x) & x > a. \end{cases} \tag{43}$$

Exercise: Use this to find the coefficients A_n, and show that they are given by

$$A_n = \frac{2hL^2}{a(L-a)}\left(\frac{1}{n^2\pi^2}\right)\sin\left(\frac{n\pi a}{L}\right). \tag{44}$$

Exercise: Using the above result, argue that if $L/a = m$, an integer, then $A_m = 0$, as is $A_{2m}, A_{3m}\ldots$

3.2.2 The struck string

We could also alter the above problem slightly to look for solutions where the string is initially at rest, but given some initial *velocity* profile (instead of a displacement profile). You should be able to convince yourself that in this case, the "boundary" conditions are:

$$\begin{aligned} u(0,t) &= 0, \\ u(L,t) &= 0, \\ u(x,0) &= 0, \end{aligned} \tag{45}$$

and the initial conditions tell us that the initial velocity is given by some velocity profile $v(x)$. That means that:

$$\frac{\partial u}{\partial t}(x,y) = v(x). \tag{46}$$

Consider a simple case where the *velocity* profile is given by the same function as before (Equation (43)), where h represents the maximum velocity, and a represents the position of this maximum.[9]

It turns out that we have very little in the way of calculations to do to solve this problem. We begin by realising that the two boundary conditions in space remain the same, and therefore so does our condition for $X(x)$. On the other hand, the "boundary" condition at $t = 0$ is different.

Exercise: Show that the fact that the string begins with no displacement profile means that

$$T(t) \sim \sin(\omega_n t). \tag{47}$$

Therefore, the general solution for a *struck* string is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(k_n x) \sin(\omega_n t). \tag{48}$$

Clearly, at $t = 0$, the displacement is zero, as expected. We now need to compute B_n, which would complete our calculations. We put in the initial conditions, which say that

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \omega_n \sin\left(\frac{n\pi x}{L}\right) = f(x). \tag{49}$$

Now, if we rename $A_n = B_n \omega_n$ (which we can do, since they're all just numbers) we find that this is just the same problem we just solved! And therefore, using Equation (44), we have:

$$B_n = \frac{A_n}{\omega_n} = \left(\frac{L}{v_p n\pi}\right) \times \frac{2hL^2}{a(L-a)} \left(\frac{1}{n^2\pi^2}\right) \sin\left(\frac{n\pi a}{L}\right). \tag{50}$$

[9] Such velocity profiles are not as unphysical as you might imagine, see Antoine Chaigne , "Reconstruction of piano hammer force from string velocity", The Journal of the Acoustical Society of America 140, 3504-3517 (2016). On a side-note, I just realised I was actually taught by the author of this paper!

3.3 A solved example

I had promised all of you that I would solve one complete initial conditon problem, chosen randomly, from your last assignment. Here is my solution, along with an associated Jupyter Notebook. The initial condition that was assigned to me is shown in Figure (2).

Figure 2: Initial condition for a plucked string: the profile follows a sine curve from 0 to $L/2$, after which it is zero. The initial velocity profile is zero. The string has both ends fixed.

Given that the string has both ends fixed, and that the initial condition is in displacement and not in time, we can say that the general solution can be written as a linear sum of standing waves of the form:[10]

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{v_p n\pi t}{L}\right). \tag{51}$$

The initial displacement profile of the string can be written as

$$f(x) = \begin{cases} h\sin\left(\frac{2\pi x}{L}\right) & 0 < x \leq L/2, \\ 0 & \text{otherwise.} \end{cases} \tag{52}$$

We would thus like to compute coefficients A_n such that

$$u(x,0) = f(x) \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x). \tag{53}$$

Using the orthogonality of the sines on the interval $(0,L)$ that we saw earlier, we can say that[11]

$$A_n = \frac{2}{L}\int_0^L \mathrm{d}x\, f(x)\sin\left(\frac{n\pi x}{L}\right). \tag{54}$$

Given the fact that our function is only non-zero in the interval 0 to $L/2$, this integral reduces to:

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^{L/2} \mathrm{d}x\, h\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right), \\ &= \frac{2h}{L}\int_0^{L/2} \mathrm{d}x\, \frac{1}{2}\left(\cos\left(\frac{(n-2)\pi x}{L}\right) - \cos\left(\frac{(n+2)\pi x}{L}\right)\right) \end{aligned} \tag{55}$$

We now have two integrals of the form:

$$I(m) = \frac{1}{L}\int_0^{L/2} \mathrm{d}x\, \cos\left(\frac{m\pi x}{L}\right), \qquad m \in \mathbb{Z}, \tag{56}$$

[10] When you write your assignment, you should start from the general solution to the wave equation and shown how this comes about, like I did in the previous section.

[11] Again, this will have to be proved by you when you write your answer.

so let us solve this for a general m. Using the fact that the integral of a cosine is a sine, we have

$$I(m) = \frac{1}{L}\frac{L}{m\pi}\sin\left(\frac{m\pi x}{L}\right)\Bigg|_{x=0}^{x=L/2} = \frac{1}{m\pi}\sin\left(\frac{m\pi}{2}\right). \quad (57)$$

Since we've seen that our integral for A_n can be written as

$$A_n = h\Big(I(n-2) - I(n+2)\Big), \quad (58)$$

we can see that

$$A_n = \frac{h}{(n-2)\pi}\sin\left(\frac{(n-2)\pi}{2}\right) - \frac{h}{(n+2)\pi}\sin\left(\frac{(n+2)\pi}{2}\right). \quad (59)$$

Here we notice something interesting: if n is an even number, then so is $n-2$ and $n+2$, and so we can write each of the sine terms as $\sim \sin(p_\pm \pi)$, where $p_\pm = (n \pm 2)/2$ is an integer. As a result, all the terms with n being even are automatically zero! Well, nearly all of them. $n = 2$ is a special case. Using L'Hôpital's rule, you should be able to show that $A_2 = 1/2$. Furthermore, the terms in the sine function are of the form $\sin(x \pm \pi) = -\sin(x)$, and so we can compute the final coefficients shown below. Figure (3) shows the results when $t = 0$.

$$\begin{aligned} A_n &= -\frac{h}{(n-2)\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{h}{(n+2)\pi}\sin\left(\frac{n\pi}{2}\right) \\ &= -\frac{h}{\pi}\left(\frac{4}{n^2-4}\right)\sin\left(\frac{n\pi}{2}\right) \end{aligned} \quad (60)$$

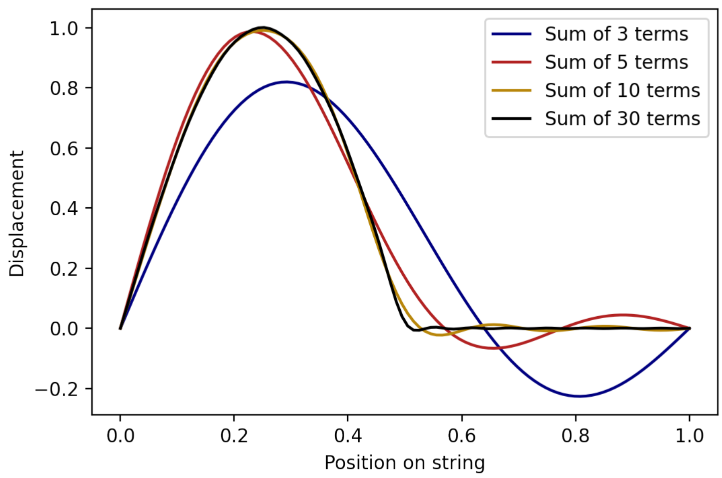


Figure 3: A plucked string with a different initial displacement profile from before: the string follows a sinusoidal function until $x = L/2$, after which it remains at equilibrium. As before, more Fourier terms leads to a better approximation of the initial function. Even when $N = 30$ (which is actually only 16 non-zero terms, as argued above) the approximation is quite good.

The code containing the animation of this string for all times $t > 0$ is attached as a Jupyter Notebook.