DS 7: Continuity and Conservation Laws

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1 Local conservation laws

A naive way to look at conservation of charge is as follows: a tiny bit of positive charge disappears from somewhere in the universe and reappears somewhere else in the universe. In this way, the *net* charge in the universe is conserved. Such a "conservation" law is called a *global* conservation law, and you would be right to doubt if it is physically acceptable. Indeed, it isn't acceptable for a very simple reason due to an important consequence of Special Relativity. In the theory of Special Relativity, you must have learnt that spatially separated events that are simultaneous in one frame of reference are not simultaneous in any other frame of reference moving relative to it. As a result, while you might see the charge disappear from one point and reappear at another instantaneously, your friend moving with respect to you would not see these two events occur simultaneously. As a result, she would believe that for some interval of time, the net charge in the universe was *not* constant, which goes against our naive law of charge conservation.

A stronger statement is that of *local* charge conservation, i.e. that the rate of change of charge in any region of space must be due to some sort of "current" moving out of that region. We will now derive the equation that governs local conservation in a very simple way, looking at the flow of water in a pipe. Imagine that you have an infinitely long pipe, through which water is flowing. You're interested in a small section of the pipe. The flow of the water is not necessarily steady: some places have it moving faster than others.

(a) Imagine a small volume of this pipe and ask yourself how the amount of water in it $Q(t)$ is related to the net "current" of water $I(t)$ that flows out of the cross-sectional area that surrounds it. Show that:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -I. \tag{1}$$

(b) This is a "global" conservation law: it relates the total amount of water at a given instant of time in a region to the total current flowing out of the region's boundary at that time. However, there is no constraint on the size of the volume. What if we made this an infinitesimally small volume? This allows us to move from "macroscopic" quantities that are defined over extended regions to "microscopic" quantities that are defined at each point. In particular, we will have to work with the *mass density* of water ρ and *current density* of water $\mathbf{j}$. Convince yourself that by definition:

(i) The total amount of water in any volume V is given by

$$Q(t) = \iiint_V \rho(x, y, z, t)\,\mathrm{d}V,$$

(ii) The total current of water flowing out of a surface S is given by

$$I(t) = \iint_S \mathbf{j}(x, y, z, t) \cdot \mathrm{d}\mathbf{A}.$$

(c) Use the divergence theorem to show that if $Q(t)$ is a *conserved* quantity, then

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{2}$$

In the above example, ρ represents the mass density of the water, but it could just as well represent any other sort of density. In particular, it could represent *charge density*, in which case $\mathbf{j}$ is the electromagnetic current density that sources magnetic fields.

2 Conservation of charge in electromagnetism

(a) Where does conservation of charge fit into our laws of electromagnetism? Is it something that must be externally imposed, or does it fall out naturally from our existent formulation? It turns out, we *don't* need to introduce it externally! In order to see this, take the divergence of the curl of $\mathbf{B}$ in Maxwell's Equations and show that you get the continuity equation for charge conservation.

(b) Now, we make things a little more interesting: you have all learnt in your course on electromagnetism that no magnetic monopoles exist, and that this is reflected in the fact that $\nabla \cdot \mathbf{B} = 0$. This is just a reflection of the fact that electric charges produce diverging electric fields and electric currents produce curly magnetic fields. Now, how would Maxwell's Equations change if such a monopole did exist? We would need both magnetic charge densities ρ_m (that produce diverging magnetic fields) and magnetic current densities (that produce curly electric fields). One convention is to use

$$\nabla \cdot \mathbf{B} = \mu_0 \rho_m. \tag{3}$$

Use this to find how the rest of Maxwell's Equations change when introducing magnetic monopoles.

3 Conservation of energy and momentum in mechanical waves

When we studied mechanical waves on a string, we saw that the vertical displacement of the string satisfies the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c_s^2} \frac{\partial^2 y}{\partial t^2}, \tag{4}$$

where c_s is the speed of the wave, and for waves on a string we saw that if we denote the tension in the string as T and the equilibrium mass density as μ,

$$c_s^2 = \frac{T}{\mu}. \tag{5}$$

In general, both the tension and the mass-density can be functions of position, so that $T \equiv T(x)$ and $\mu \equiv \mu(x)$, but in this problem we will simplify things by assuming that the string is homogeneous with a constant mass density, and ignore situations where T varies with the position in the medium (such as a rope that hangs vertically in a gravitational field). As a result, in what follows, we take c_s to be a constant.

(a) In class we saw that the energy density for such a mechanical wave is given by

$$u_E = \frac{1}{2}\mu \left[\left(\frac{\partial y}{\partial t} \right)^2 + \left(c_s \frac{\partial y}{\partial x} \right)^2 \right] \tag{6}$$

Show that

$$\frac{\partial u_E}{\partial t} = \mu c_s^2 \left(\frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x \partial t} \right) \tag{7}$$

(b) Use the chain rule on the second term on the right-hand side above to show that you can define an *energy-current* j_E

$$j_E = -T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}, \tag{8}$$

such that the energy density and energy current satisfy a (one-dimensional) continuity equation:

$$\frac{\partial u_E}{\partial t} + \frac{\partial j_E}{\partial x} = 0. \tag{9}$$

(c) Similarly, one could define a *momentum*-density

$$u_P = \mu \frac{\partial y}{\partial t}. \tag{10}$$

Use this to show that the momentum also satisfies a continuity equation

$$\frac{\partial u_P}{\partial t} + \frac{\partial j_P}{\partial x} = 0. \tag{11}$$

and that the corresponding momentum-current j_P is

$$j_P = -T \frac{\partial y}{\partial x}. \tag{12}$$