# DS 8: Mock Examination

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(a) **Question 1: Forced Oscillations** **[15]**

(i) Find the general solution to the the differential equation

$$\frac{d^3x}{dt^3} + x(t) = 0. \tag{1}$$

(ii) Now, introduce a forcing term and find the general solution, i.e. solve

$$\frac{d^3x}{dt^3} + x(t) = \cos(\omega t). \tag{2}$$

(b) **Question 2: Systems of Coupled Oscillators** **[20]**

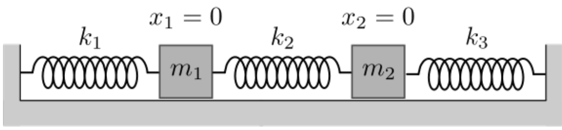


**Figure 1:** Two masses $m_1 = 4$ kg and $m_2 = 1$ kg are placed between three springs of constants $k_1 = 64$ Nm$^{-1}$, $k_2 = 16$ Nm$^{-1}$, $k_3 = 4$ Nm$^{-1}$, and the system is allowed to oscillate.

(i) Consider the system described in Figure (1), with $m_1 = 4$ kg and $m_2 = 1$ kg, and spring constants are $k_1 = 64$ Nm$^{-1}$, $k_2 = 16$ Nm$^{-1}$, $k_3 = 4$ Nm$^{-1}$. Find the normal mode frequencies.

(ii) Without calculating the normal modes, which of the two frequencies do you think corresponds to the in-phase, and which correspond to the out-of-phase motion of the system? Why?

(iii) Now, compute the normal modes and see if your intuition is correct.

(c) **Question 3: Parseval's Theorem** **[15]**

(i) Given that a function of period $2L$ can be represented by a Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right), \tag{3}$$

prove the important result known as Parseval's theorem, which states that:

$$\frac{1}{L}\int_{-L}^{L} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \tag{4}$$

(ii) Can you explain why this is the generalisation of the Pythagorean theorem for Fourier Series?

(d) **Bonus: The beaded string** **[Bonus 20]**

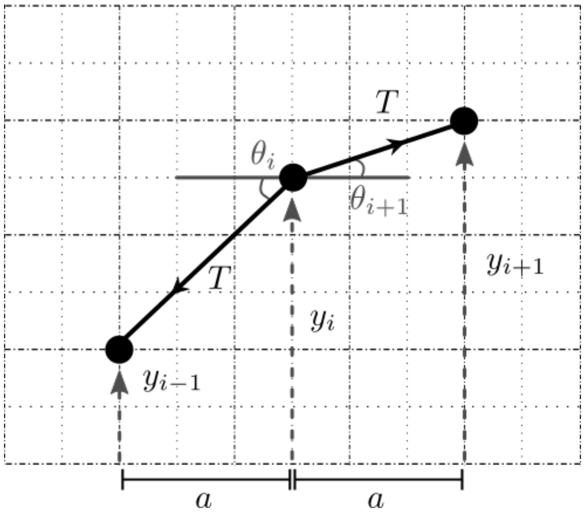


**Figure 2:** A section of a beaded string: the masses, each with mass $m_i = m$ and with coordinates $x_i$ and $y_i$, are spaced a distance $a$ apart and connected by a massless string, held together by some tension $T$. We assume small oscillations, meaning that the angles $|y_i| \ll a$, and so $\theta_i \ll 1$.

(i) Consider a beaded string, as shown in Figure (2). Show by applying Newton's laws to the bead $i$ that when there is no horizontal displacement, the equations of motion in the $y$ direction is:

$$m\ddot{y}_i = -T\sin\theta_i + T\sin\theta_{i+1}, \tag{5}$$

(ii) Use the small-angle approximation to show that this equation can be written as

$$\ddot{y}_i = \omega_0^2\left(y_{i-1} - 2y_i + y_{i+1}\right). \tag{6}$$

(iii) Suppose there are $N = 3$ beads. Show that the physics of the left-hand wall can be incorporated by going to an infinite system, and requiring the boundary condition $y_0 = 0$. Find the boundary condition for $y_4$.

(iv) Use a trial solution of the form $y_i = A\sin(kx_i)\cos(\omega t)$. First use it with the boundary conditions to arrive at the allowed values of $k_n$. Then show that

$$\omega_n = 2\omega_0 \sin\left(\frac{k_n a}{2}\right). \tag{7}$$