DS 9:
Energy Conservation in Electromagnetism

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1 The Poynting Vector

We saw in class how the total energy in an electromagnetic field can be written as

$$U_{\text{em}} = \iiint_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) \mathrm{d}V. \tag{1}$$

From this, it's clear that the energy *density* in the electromagnetic field is given by

$$u_{\text{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right). \tag{2}$$

We will now try to show this more generally.

(a) Consider a distribution of charges and currents that – at some instant of time – produces electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. In addition, the fields interact with these charges and do work on them. The work done by the electric fields on the charges is given by

$$\mathrm{d}W = \mathbf{F}\cdot \mathrm{d}\mathbf{l}. \tag{3}$$

Use the Lorentz Force Law to show that

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \iiint_V \mathbf{E}\cdot\mathbf{j}\,\mathrm{d}V, \tag{4}$$

where $\mathbf{j} = \rho\mathbf{v}$ is the current density associated with a charge density ρ moving at velocity $\mathbf{v}$.

(b) Use Maxwell's Equations to rewrite $\mathbf{j}$ in terms of (derivatives of) the electric and magnetic fields

$$\mathbf{E}.\mathbf{j} = \frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \epsilon_0\mathbf{E}\cdot\frac{\partial E}{\partial t}. \tag{5}$$

(c) From vector calculus, we know that

$$\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B}). \tag{6}$$

Using this relation and Maxwell's Equations, show that in terms of $E^2 \equiv |\mathbf{E}|^2$ and $B^2 \equiv |\mathbf{B}|^2$:

$$\mathbf{E}.\mathbf{j} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) - \frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B}). \tag{7}$$

(d) From the above results, we can now prove **Poynting's Theorem**:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\iiint_V \frac{\partial}{\partial t}\left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right)\mathrm{d}V - \oiint_S (\mathbf{E}\times\mathbf{B})\cdot \mathrm{d}\mathbf{S}. \tag{8}$$

This theorem states that the work done on the system of charges by the electromagnetic fields in a volume is equal to the decrease in energy stored in the field in that volume, minus the energy that flowed out through the surface bounding the volume.

(e) Write the above equation in terms of the energy density u_{em} and the energy *flux* density, called the Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}), \tag{9}$$

and show that

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\iiint_V \left(\frac{\partial u_{\mathrm{em}}}{\partial t} + \nabla\cdot\mathbf{S}\right)\mathrm{d}V. \tag{10}$$

(f) Now, W is the work done on the system of charges, and it will go into changing their mechanical energy. As a result, we can write W in terms of a mechanical energy density:

$$W = \iiint_V u_{\mathrm{mech}}\,\mathrm{d}V. \tag{11}$$

Use the above result to show that we get a continuity equation for the energy of the system:

$$\frac{\partial}{\partial t}(u_{\mathrm{em}} + u_{\mathrm{mech}}) + \nabla\cdot\mathbf{S} = 0. \tag{12}$$

Here, $u_{\mathrm{em}} + u_{\mathrm{mech}}$ is the net energy density (both mechanical and electromagnetic) and $\mathbf{S}$ is the energy flux density, which represents the *flow* of energy, just like $\mathbf{j}$ represents the flow of charge.

2 An analogy with mechanical waves

Let us consider a system without any sources (i.e. in free space) so that there are only electromagnetic fields. In this case, $u_{\mathrm{mech}} = 0$, and we have a continuity equation between

$$\begin{aligned} u_{\mathrm{em}} &= \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right), \\ \mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}). \end{aligned} \tag{13}$$

In a previous DS, we saw that energy conservation in mechanical waves involved a continuity equation between the energy density and energy current

$$\begin{aligned} u_E &= \frac{1}{2}\mu\left[\left(\frac{\partial y}{\partial t}\right)^2 + \left(c_s\frac{\partial y}{\partial x}\right)^2\right], \\ j_E &= -T\,\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}. \end{aligned} \tag{14}$$

At first inspection, the Equations (14) and (13) do not seem to have much in common at all. The mechanical expressions are written in terms of derivatives of the displacement y, while the electromagnetic quantities are written in terms of the fields. It turns out that there is in fact a way to make these two equations resemble each other more closely, using the vector potential $\mathbf{A}$.

(a) Show that, in the absence of free charges and currents, we can write

$$\begin{aligned} \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t}, \\ \mathbf{B} &= \nabla \times \mathbf{A}. \end{aligned} \tag{15}$$

Hint: You will need to make a restricted gauge choice for this to work. Luckily, gauge invariance allows us to make such a choice. Find out what this specific gauge is called.

(b) Using this definition, write out u_{em} and $\mathbf{S}$ in terms of $\mathbf{A}$.

(c) Let us now work with electromagnetic plane waves, moving along the $\mathbf{k} \equiv \hat{\mathbf{x}}$. We choose the electric field along $\hat{\mathbf{y}}$ and the magnetic field along $\hat{\mathbf{z}}$. If we choose

$$\begin{aligned} \mathbf{E}(x,t) &= E_0 \cos\big(kx - \omega t + \phi\big)\hat{\mathbf{y}}, \\ \mathbf{E}(x,t) &= \frac{E_0}{c} \cos\big(kx - \omega t + \phi\big)\hat{\mathbf{z}}. \end{aligned} \tag{16}$$

Show that the vector potential corresponding to this configuration is:

$$\mathbf{A}(x,t) = \frac{E_0}{\omega} \sin\big(kx - \omega t + \phi\big)\hat{\mathbf{y}} = A_y(x,t)\hat{\mathbf{y}} \tag{17}$$

(d) Use the above vector potential in the equations for the energy density and Poynting vector to show that

$$\begin{aligned} E^2 &= \left(\frac{\partial \mathbf{A}}{\partial t}\right)^2 = \left(\frac{\partial A_y}{\partial t}\right)^2, \\ B^2 &= (\nabla \times \mathbf{A})^2 = \left(\frac{\partial A_y}{\partial x}\right)^2, \\ \mathbf{E} \times \mathbf{B} &= \frac{\partial \mathbf{A}}{\partial t} \times (\nabla \times \mathbf{A}) = \frac{\partial A_y}{\partial t}\frac{\partial A_y}{\partial x}\hat{\mathbf{x}}. \end{aligned} \tag{18}$$

Use these results to show that

$$\begin{aligned} u_{\text{em}} &= \frac{1}{2}\epsilon_0 \left[\left(\frac{\partial A_y}{\partial t}\right)^2 + \left(c\frac{\partial A_y}{\partial x}\right)^2\right], \\ \mathbf{S} &= -\frac{1}{\mu_0}\frac{\partial A_y}{\partial x}\frac{\partial A_y}{\partial t}. \end{aligned} \tag{19}$$

In other words, we get equations that are essentially identical to Equation (14), with c_s being replaced by the speed of light c, by making the association:

$$\begin{aligned} y &\leftrightarrow A_y, \\ \mu &\leftrightarrow \epsilon_0, \\ T &\leftrightarrow \frac{1}{\mu_0}. \end{aligned} \tag{20}$$