

# DS 11: Motivating the Fourier Transform

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## 1 The complex Fourier Series representation

From our earlier classes, you should remember that if a function is periodic over some period  $T_0$ , i.e.  $f(t + T_0) = f(t)$ , then  $f(t)$  could equivalently be represented by an infinite series of sines and cosines, called a **Fourier Series**:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(2\pi n \frac{t}{T_0}\right) + b_n \sin\left(2\pi n \frac{t}{T_0}\right) \right) \quad (1)$$

An equivalent, though somewhat neater, way to write this series is to use the fact that the sines and cosines can be expressed in terms of complex exponentials.

**Exercise:** Show that you can write

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega_0 t}, \quad (2)$$

where  $\omega_0 = 2\pi/T_0$ , and  $c_n$  is some coefficient you need to find in terms of  $a_n$  and  $b_n$ .

**Exercise:** Show that  $c_{-n} = c_n^*$ .

**Exercise:** Can you now see why in the original definition of the Fourier Series the constant term had an extra factor of  $1/2$  that the other terms did not have?

Notice how all the information contained in the function  $f(t)$ , or rather, all the information required to reconstruct the function  $f(t)$  is present in the infinite (but *countably* infinite) set of coefficients  $\{c_n\}$ . The components  $c_n$  tell us how much of the different frequencies (all of which are some integral multiple of  $\omega_0$ ) are present in the function  $f(t)$ . Thus, we can say that

$$\begin{aligned} f(t) &\longrightarrow \text{contains information in the "time" domain,} \\ c_n &\longrightarrow \text{contains information in the "frequency" domain,} \end{aligned}$$

The coefficients  $c_n$  can be obtained from the function  $f(t)$  by performing the integral:

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{in\omega_0 t} dt. \quad (3)$$

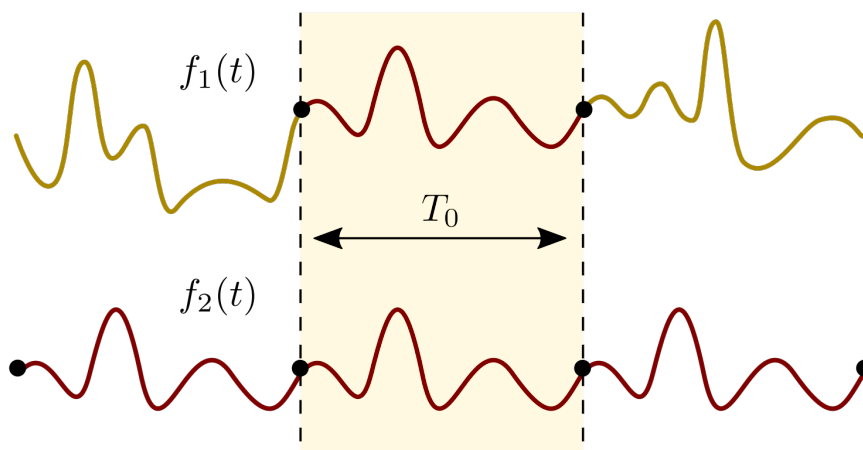
These are obtained in more or less exactly the same way that the coefficients  $a_n$  and  $b_n$  are obtained in the usual definition of the Fourier Series.

**Exercise:** Show that the functions  $e^{in\omega_0 t}$  are orthogonal over a period of  $T_0 = 2\pi/\omega_0$ . i.e., show that:

$$\int_{-T_0/2}^{T_0/2} e^{im\omega_0 t} e^{in\omega_0 t} dt = T_0 \delta_{mn}. \quad (4)$$

## 2 Extending our definition to non-periodic functions

Notice how the above definition only works for periodic functions. Suppose, instead, that we don't have a periodic function, but we are only interested in the function between two points, say,  $(-T_0/2, T_0/2)$ . We can use a trick: the idea is that we imagine that the function is periodic, with period  $T_0$ . Notice how it's not the same function, but both functions overlap over the interval of interest. We have thus constructed a periodic function over all space, but we're only interested in a small part, from  $-T_0/2$  to  $T_0/2$ . We can of course get the Fourier components for this function, provided we understand that they are only to be used to approximate the function  $f(t)$  in the appropriate regime.



**Figure 1:** Two functions that are not equal in general, but are equal in an interval of width  $T_0$ .

Thus, over the interval  $T_0$ , we can express  $f(t)$  as a sum over sinusoidal functions with frequencies that are integral multiples of a smallest frequency  $\omega_0$ , when

$$\omega_0 = \frac{2\pi}{T_0}. \quad (5)$$

Of course, since the function is inherently non-periodic, there is nothing special about the value of  $T_0$  that we chose. We could always have defined a different interval (say,  $T_1 > T_0$ ). In this way, we could re-express  $f(t)$  between  $-T_1/2$  and  $T_1/2$  as a combination of sinusoidal functions with a smaller fundamental frequency,

$$\omega_1 = \frac{2\pi}{T_1}. \quad (6)$$

But can we use this technique to completely describe an arbitrarily long signal, i.e. a purely non-periodic function. We can think of such a function as have a period of  $T_0 \rightarrow \infty$ . We will need to take this limit

very carefully, to remain consistent. First, we notice that as  $T_0$  increases, the fundamental frequency  $\omega_0$  decreases. Indeed, this is a reflection of the fact that the spacing between the different frequencies required to describe the function goes to zero. Therefore, the discrete variable  $\omega_n = n\omega_0$  now goes to a *continuous* variable,  $\omega$ , since the separation between subsequent values goes to zero. Furthermore, since the frequencies  $\omega_n$  are now continuous, the discrete Fourier coefficients must *also* be continuous. Thus, to summarise, in this limit:

$$\begin{array}{ll} T_0 \rightarrow \infty & \text{(an aperiodic function has an infinite period)} \\ \Delta\omega \equiv \omega_0 \rightarrow d\omega & \text{(an infinitesimal quantity)} \\ n\Delta\omega \rightarrow \omega & \text{(a continuous variable)} \\ c_n \rightarrow c(\omega) & \text{(a function of a continuous variable)} \end{array}$$

Given our discussion above, we would expect that the discrete Fourier sum must now be replaced by an integral. We can do this by writing  $f(t)$  as a limit of a sum, and taking the limit  $T_0 \rightarrow \infty$ , i.e.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega_0 t} = \frac{1}{\Delta\omega} \sum_{n=-\infty}^{\infty} c_n e^{-in\omega_0 t} \Delta\omega.$$

Using the above scheme, we now have two equations for  $f(t)$  and  $c(\omega)$  that look more symmetric:

$$\begin{aligned} f(t) &= \frac{1}{\Delta\omega} \sum_{n=-\infty}^{\infty} c_n e^{-in\omega_0 t} \Delta\omega & \xrightarrow{T_0 \rightarrow \infty} & \frac{1}{\Delta\omega} \int_{-\infty}^{\infty} c(\omega) e^{-i\omega t} d\omega, \\ c(\omega) &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{in\omega_0 t} dt & \xrightarrow{T_0 \rightarrow \infty} & \frac{1}{T_0} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt. \end{aligned} \tag{7}$$

However, the astute among you would be right to be concerned: the above equations do look a little bogus; the biggest problems with them have been highlighted in red: the constant terms in front of the integrals both behave undesirably as  $T_0 \rightarrow \infty$ . In the first case,  $\Delta\omega \rightarrow 0$ , as so the constant term blows up, while in the second case, the  $1/T_0$  term takes the right-hand-side to 0. So while this idea nearly seems to work, it doesn't *quite* work yet. The trick is to realise that while  $\Delta\omega$  and  $T_0$  go to 0 and  $\infty$  respectively, they do so at the same rate, such that  $T_0 \times \Delta\omega = 2\pi$ . This seems to indicate that  $c(\omega)$  might not be the ideal function to work with when  $\Delta\omega \rightarrow 0$ . Thus, the solution is to redefine things and work instead with

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt. \tag{8}$$

**Exercise:** Using the above definition, show that  $f(t)$  can be written in terms of  $F(\omega)$  as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega. \tag{9}$$

As you can see – using  $F(\omega)$  – all the undesirable behaviour of both functions seem to have been dealt with. The function  $F(\omega)$  is called the **Fourier Transform** of  $f(t)$ : it takes  $f(t)$  – a function in the “time” domain – and produces a function in the *frequency* domain  $\omega$ . You should, in a coarse sense, consider  $F(\omega)$  to play the same role that the Fourier Coefficients did in the case of periodic functions: it is a measure of

“how much” of the frequency  $\omega$  the function is composed of. Similarly,  $f(t)$  is called the *inverse* Fourier Transform of  $F(\omega)$ .

Now, for all of this to make any sense at all, we require that the inverse Fourier Transform of the Fourier Transform of  $f(t)$  just give back  $f(t)$  again, since this is just a basic consistency requirement. Thus, we require

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \underbrace{\int_{-\infty}^{\infty} dt' f(t') e^{-i\omega t'}}_{F(\omega)}. \quad (10)$$

The above condition is known as the Fourier Integral Theorem. We can prove it as follows: we first rearrange the terms in this integral (since we’re assuming everything converges, and so on<sup>1</sup>) and get

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' f(t') \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')}. \quad (11)$$

Now, we perform the second integral on the right-hand side. In order to do this, we will need a result that you will prove in this week’s assignment:

$$\int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')} = 2\pi\delta(t-t'), \quad (12)$$

where  $\delta(\tau)$  is the Dirac Delta function. As a result, we can show that Equation (11) is indeed consistent, since

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' f(t') \times 2\pi\delta(t-t') = f(t). \quad (13)$$

### 3 Problems:

- (a) Prove the Fourier Shift Theorem, i.e. if we denote the Fourier Transform of a function  $f(t)$  by  $\mathcal{F}[f(t)](\omega)$ , prove that

$$\mathcal{F}[f(t-t_0)](\omega) = e^{i\omega t_0} \mathcal{F}[f(t)](\omega) = e^{i\omega t_0} F(\omega). \quad (14)$$

- (b) Prove the Convolution Theorem. Defining the convolution of two functions  $f(t)$  and  $g(t)$  as being

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t)g(u-t)dt, \quad (15)$$

show that:

$$\mathcal{F}[f(t) * g(t)] = F(\omega)G(\omega). \quad (16)$$

- (c) Prove Parseval’s Theorem for Fourier Transforms:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \quad (17)$$

- (d) Prove the Differentiation Theorem for the Fourier Transform

$$\mathcal{F}[f'(t)] = -i\omega F(\omega), \quad (18)$$

and generalise this to find  $\mathcal{F}[f^{(n)}(t)]$  in terms of  $F(\omega)$ .

<sup>1</sup>The more mathematically minded of you would be right to be wary of the reckless abandon with which this was done. I’m sure there are many mathematical subtleties that I am sweeping under the rug, but they are not pertinent to our discussion here.