DS 1:
Partial Derivatives

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1 The reciprocal and reciprocity theorems

Consider a function f of two variables (say, x and y). By definition, a small change in the function can be written in terms of the partial derivatives of f with respect to x and y, in each case keeping the other variable constant. Therefore, we can write

$$\mathrm{d}f = \left(\frac{\partial f}{\partial x}\right)_y \mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)_x \mathrm{d}y \tag{1}$$

The above relation is the most important mathematical relation concerning functions of more than one variable. It will be essential in this course, and we will use it repeatedly. In this session, we will use the above equation to derive two more very important relations.

(a) Divide Equation (1) (i) by $\mathrm{d}x$ while holding y constant, and (ii) by $\mathrm{d}y$ while holding x. You should not be too surprised by your result.

(b) Now, do something more ambitious: divide Equation (1) by $\mathrm{d}f$, keeping y constant, and use it to show our first important result: the "reciprocal" theorem.

The reciprocal theorem:

$$\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial x}{\partial f}\right)_y^{-1}. \tag{2}$$

(c) Verify this theorem using the function

$$f(x, y) = 2y \ln x + (y-1)^2. \tag{3}$$

(d) Now, divide Equation (1) by $\mathrm{d}x$ while holding f constant. Use this to show our second important result: the "reciprocity" theorem.

The reciprocity theorem:

$$\left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_f \left(\frac{\partial x}{\partial f}\right)_y = -1. \tag{4}$$

Notice how, unlike with the reciprocal theorem, you would have got the answer if you had treated partial derivatives like "normal" first-order derivatives without paying attention to what is being kept constant.

2 Proper and improper differentials

In this question, we will deal with what it means for a function f to be "well-behaved" in Physics. For now, let us say, for simplicity, that a well-behaved function is one that can be written as a power series involving all powers of x and y, and any combinations of the two. In other words, a "well-behaved" function is one that can be written as

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n,m} x^n y^m. \tag{5}$$

(a) Show, using the definition above, that this implies that the *mixed* partial derivatives of f are equal. In other words, that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}. \tag{6}$$

From now on, we will assume this relation.

(b) Let's consider the following infinitesimal quantity

$$\mathrm{d}w = x^2 \mathrm{d}x + 2xy\mathrm{d}y. \tag{7}$$

Compute the mixed partial derivatives, and show that

$$\partial^2 w/\partial x \partial y \neq \partial^2 w/\partial y \partial x\,. \tag{8}$$

(c) Argue that there is no function w such that $\mathrm{d}w = x^2\mathrm{d}x + 2xy\mathrm{d}y$.

However, the quantity $x^2\mathrm{d}x + 2xy\mathrm{d}y$ is well defined, and is an "infinitesimal" quantity, so we should have some notation for it. The standard notation is to use a "bar" over the d.

$$đw = x^2 \mathrm{d}x + 2xy\mathrm{d}y. \tag{9}$$

(d) To show that this differential is not exact, calculate the integral of $\mathrm{d}w$ from $(x,y) = (0,0)$ to $(x,y) = (1,1)$ in two ways: (i) along the path from $(0,0)$ to $(0,1)$ to $(1,1)$, and (ii) along the diagonal line $y = x$.

(e) Now, consider the quantity

$$\frac{đw}{x} = x\mathrm{d}x + 2y\mathrm{d}y. \tag{10}$$

Show that it is an exact differential. Thus, the factor x^{-1} converts an "inexact" differential to an "exact" differential! Such factors are known as *integrating factors*.

(f) Using the above result, comment on the equation that we saw in class today

$$đQ = T\mathrm{d}S. \tag{11}$$