DS 3:
Adiabatic Processes and Response Functions

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1 Adiabats for an ideal gas

(a) An adiabatic process is one in which no heat is transferred between the system and its surroundings. Use this to show that for such a process

$$\mathrm{d}U = -P\mathrm{d}V. \tag{1}$$

(b) The above result is completely general. Now, we will restrict ourselves to a specific system, the *ideal gas*. Using the two definitions of the ideal gas (i) that $PV = Nk_BT$ and (ii) that U is independent of P, show that

$$P\mathrm{d}V + V\mathrm{d}P = -\left(\frac{Nk_BP}{C_V}\right)\mathrm{d}V. \tag{2}$$

(c) Rearrange the above equation and integrate it, using the fact that for an ideal gas

$$C_P - C_V = Nk_B, \tag{3}$$

and show that for an ideal gas, PV^γ = constant. What is the value of γ?

(d) Use the above result to show that for an ideal gas, the equation of the adiabats could just as well be written as

$$TV^{\gamma-1} = \text{constant} \qquad \text{or} \qquad P^{1-\gamma}T^\gamma = \text{constant}. \tag{4}$$

(e) Which of the two curves is steeper, the "adiabat", or the "isotherm"?

2 Adiabats for the van der Waals gas

(a) Let us now consider a monoatomic van der Waals gas. For such a gas, the equation of state and the internal energy are given by

$$\left(P + a\frac{N^2}{V^2}\right)(V - Nb) = Nk_BT, \qquad \text{and} \qquad U = \frac{3}{2}Nk_BT - a\frac{N^2}{V}. \tag{5}$$

For such a gas, show that

$$\mathrm{d}U = C_V\mathrm{d}T + a\left(\frac{N^2}{V^2}\right)\mathrm{d}V. \tag{6}$$

(b) Suppose you're moving along an adiabat. Show that the quantity

$$(V - Nb)T^{C_V/Nk_B} = \text{constant}. \tag{7}$$

(c) Verify that the result gives back the ideal gas result when $a, b \to 0$.

3 Response functions

(a) Of the response functions described below, which are intensive and which are extensive?

$$\begin{aligned}
\kappa_T &= -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T, && \text{isothermal compressibility,} \\
\kappa_S &= -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_S, && \text{adiabatic compressibility,} \\
B_T &= \frac{1}{\kappa_T}, && \text{isothermal bulk modulus,} \\
B_S &= \frac{1}{\kappa_S}, && \text{adiabatic bulk modulus,} \\
\alpha &= \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P, && \text{isobaric expansivity,} \\
C_V &= \left(\frac{\partial Q}{\partial T}\right)_V, && \text{heat capacity at constant volume,} \\
C_P &= \left(\frac{\partial Q}{\partial T}\right)_P, && \text{heat capacity at constant pressure.}
\end{aligned} \tag{8}$$

(b) Compute κ_T and α for a van der Waals gas.

(c) Show that for a van der Waals gas,

$$\left(\frac{\partial C_V}{\partial V}\right)_T = 0, \qquad \text{and} \qquad C_P - C_V = \frac{Nk_B V^3 (V - Nb)^2)}{Nk_B V^3 - 2a(V - Nb)^2}. \tag{9}$$