DS 3: Linear Vector Spaces

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1 Commuting Operators and Simultaneous Eigenstates

(a) Consider the following matrices:

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \hat{B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{1}$$

(i) Show that $\hat{A}$ and $\hat{B}$ commute.

(ii) Find the eigenvalues and corresponding (normalised) eigenvectors of the matrix $\hat{A}$.

(iii) Show that these eigenvectors are also eigenvectors of $\hat{B}$. What are their corresponding eigenvalues?

(b) Consider the following matrices:

$$\hat{P} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \hat{Q} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{2}$$

(i) Find the eigenvalues and eigenvectors of $\hat{P}$.

(ii) Show that $\hat{P}$ has a degenerate subspace.

(iii) Does $\hat{Q}$ resolve this degeneracy?

2 Two-dimensional vector spaces

Suppose we have a two dimensional Hilbert space with basis $|1\rangle$ and $|2\rangle$, where $\langle 1|2\rangle = \langle 2|1\rangle = 0$ and $\langle 1|1\rangle = 1 = \langle 2|2\rangle$. Imagine an operator $\hat{A}$, such that $\hat{A}|1\rangle = -i|2\rangle$ and $\hat{A}|2\rangle = i|1\rangle$. Suppose now we have two states:

$$|\psi\rangle = |1\rangle + 2|2\rangle$$
$$|\phi\rangle = |1\rangle + i|2\rangle$$

(a) Calculate $\langle\phi|\psi\rangle$ directly.

(b) Represent the states as column vectors $\underline{\psi}$ and $\underline{\phi}$, and verify that $\langle\phi|\psi\rangle = \underline{\phi}^\dagger\underline{\psi}$.

(c) Calculate $\hat{A}|\psi\rangle$ and $\hat{A}|\phi\rangle$ directly.

(d) Do the same thing with matrices.

(e) Calculate $\langle\phi|\hat{A}|\psi\rangle$ and $\langle\psi|\hat{A}|\phi\rangle$ directly and using matrices.

(f) If

$$|a\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}$$
$$|b\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}$$

Verify that this is an orthonormal basis.

(g) Find, in this new basis, $\underline{\psi}'$, $\underline{\phi}'$, $\hat{A}'$, and verify that

$$\underline{\psi}'^{\dagger}\underline{\phi}' = \underline{\psi}^{\dagger}\underline{\phi}$$
$$\underline{\phi}'^{\dagger}\underline{\psi}' = \underline{\phi}^{\dagger}\underline{\psi}$$
$$\underline{\psi}'^{\dagger}\hat{A}'\underline{\phi}' = \underline{\psi}^{\dagger}\hat{A}\underline{\phi}$$