DS 4: Linear Vector Spaces

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1 Commutators

The commutator is defined as

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

Commutators can be a little tricky to deal with for complicated functions of operators (the Assignment should have proved that to you!), so letâĂŹs have a look at some different formulas we can use to make manipulating them a little easier. However, only the above definition is fundamental.

Using this definition, show each of these formulae:

(a) $[\hat{A}, c] = 0$

(b) $[\hat{A}, \hat{A}] = 0$

(c) $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$

(d) $[c\hat{A}, \hat{B}] = [\hat{A}, c\hat{B}] = c[\hat{A}, \hat{B}]$

(e) $[\hat{A}, \hat{B} \pm \hat{C}] = [\hat{A}, \hat{B}] \pm [\hat{A}, \hat{C}]$

(f) $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

(g) $[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$

(h) $[\hat{A}, f(\hat{A})] = 0$

2 The Dirac Delta Function

The defining feature of the Dirac Delta Function is the following

$$f(x_0) = \int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x - x_0) f(x)$$

with $f(x)$ as any function. It should be clear that $\delta(x)$ is not a function in the usual sense. The Delta function is an example of a *distribution*, or a generalised function. In particular, it has the property that $\delta(0) = 0$ for $x \neq 0$ and $\delta(0) = \infty$. Nevertheless, it is possible (with some care) to treat it as if it were a function, while keeping in mind that it is fundamentally a "distribution", a fundamentally different object that lives in the shade of an implied integral sign.

Using the above definition, show that:

(a) $\delta(-x) = \delta(x)$

(b) $x\delta(x) = 0$

(c) $\delta(ax) = \frac{1}{|a|}\delta(x)$

(d) $\delta(x^2 - a^2) = \frac{1}{2|a|}\left(\delta(x-a) + \delta(x+a)\right)$

(e) $\int \mathrm{d}x\, \delta(a-x)\delta(x-b) = \delta(a-b)$

(f) $f(x)\delta(x-a) = f(a)\delta(x-a)$

To prove some of these identities, it may be useful to remember that the $\delta-$function is closely related to the Heaviside step function

$$\theta(x-a) = \int_{-\infty}^{x} \delta(y-a)\,\mathrm{d}y$$
$$\Updownarrow$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\theta(x-a) = \delta(x-a)$$

The derivative of the Dirac Delta

The derivative of the Dirac Delta is again defined by its action on functions (just like the $\delta-$function itself).

(a) Calculate

$$\int_{-\infty}^{\infty} \delta'(x) f(x)\,\mathrm{d}x$$

(b) Show that

$$x\delta'(x) = -\delta(x)$$

(c) Show that

$$\delta'(ax) = \frac{1}{|a|^2}\delta'(x)$$