DS 5:
The Schrodinger Equation

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September 25, 2019

1 Position and momentum as differential operators

In class, you should have seen that the states of definite position $|x\rangle$ can be represented by δ−functions in position space. From this, you could show that

$$\langle x|p\rangle = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} \tag{1}$$

You have seen that the action of the position operator ($\hat{x}$) on a state is equivalent to multiplying the (position) wavefunction by x. In class, you saw that the action of the momentum operator ($\hat{p}$) on a state is equivalent to acting the differential operator $-i\hbar\hat{D}$ on the (position) wavefunction. i.e.

$$\langle x|\hat{p}|\psi\rangle = \frac{\hbar}{i}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}$$

(a) Consider the action of $\hat{p}$ on an arbitrary state $|\varphi\rangle$. What is the equivalent wavefunction in **momentum** space? i.e. what is

$$\langle p|\hat{p}|\varphi\rangle = ?$$

(b) Now consider the action of the position operator $\hat{x}$ on an arbitrary state $|\varphi\rangle$. What is the equivalent wavefunction in **momentum** space?

(c) Interpret the following quantities physically, and evaluate them first in terms of $\varphi(x)$ and $\phi(x)$, and then in terms of $\tilde{\varphi}(p)$ and $\tilde{\phi}(p)$.

$$\langle\varphi|\hat{x}|\phi\rangle = ?$$
$$\langle\varphi|\hat{p}|\phi\rangle = ?$$

2 Time-evolution of states of definite position

Consider a system prepared initially in a state of definite position $|x\rangle$. Let us now ask ourselves how the system behaves as time passes. You have seen that the time-evolution of a quantum system depends on the Hamiltonian $\hat{H}$ of the system. If the system is prepared in an eigenstate of the Hamiltonian, then its state does not change with the passage of time.[1]

[1] Except, perhaps, for an overall phase factor which is generally not physically relevant.

(a) If a system is prepared in a state $|x\rangle$, will it continue to remain in that same state, or will it change? Explain your answer.

(b) Let us now try to see if your answer is correct. Convince yourself that if a system is initially prepared in some state $|\psi\rangle$ at $t = 0$, and you wish to know the wavefunction at some later time t, then the wavefunction

$$\psi(x,t) = \langle x|\psi(t)\rangle = \langle x|e^{-i\frac{\hat{H}}{\hbar}t}|\psi(0)\rangle$$

(c) Now, convince yourself that – if the state is initially a state of definite position (say $|x'\rangle$, then you are interested in finding

$$\psi(x,t) = \langle x|e^{-i\frac{\hat{H}}{\hbar}t}|x'\rangle$$

(d) Compute the above quantity, and show that

$$\psi(x,t) = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left\{\frac{im\left(x-x'\right)^2}{2\hbar t}\right\}$$

(e) Compute $|\psi(x,t)|^2$. Do you see a problem with this? Interpret your result physically. Is this what you expected to happen?

(f) It turns out that this initial condition is probably not physical. A more "realistic" initial condition would be to imagine a highly peaked Gaussian distribution in x, and allow *that* to evolve in time, as you will do in Assignment 2.

(g) Repeat the above calculation if the system was initially prepared in a state of definite momentum $|p\rangle$. Does *this* agree with what you would expect?[2]

3 Recognising and old friend

This should surprise you. Perform the following operations, and be stunned:

(a) Write out the Schrodinger Equation.

(b) Write out the complex-conjugate of the Schrodinger Equation. (Convince yourself – hopefully quickly – that these are not two *different* equations).

(c) Multiply the first by $\psi^*(x)$ and the second by $\psi(x)$ and subtract them. Show that you get:

$$\begin{aligned} i\hbar\frac{\partial\psi^*\psi}{\partial t} &= -\frac{\hbar^2}{2m}\left(\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right) \\ &= -\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right) \end{aligned}$$

[2] Can you draw a parallel with Classical Mechanics here?

(d) Rewrite the equation in terms of two new quantities:

$$\rho = \psi^* \psi$$
$$j = \frac{\hbar}{2im}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$$

(e) Generalise – with reckless abandon – to three dimensions, and recognise an old friend.