DS 7:
Wave Mechanics

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1 The Quantum Bouncer

Consider a quantum particle experiencing a constant force F in (say) the $x-$direction.

(a) Write out the time-independent Schrodinger Equation satisfied by this particle.

(b) Suppose $\psi_0(x)$ is an allowed energy level, with corresponding eigenvalue E_0. Show that if *one* value of E_0 is allowed, *any* other E (positive or negative) is allowed. How are their eigenstates related?

(c) Choose a convenient value of E_0, and solve the equation.

2 The Harmonic Oscillator

(a) A harmonic oscillator is in a state such that a measurement of the energy yields either $(1/2)\hbar\omega$ or $(3/2)\hbar\omega$ with equal probability. What is the largest possible value of $\langle p \rangle$ in such a state? If it assumes this maximal value at $t = 0$, what is $\Psi(x, t)$?

(b) Let us now relax the condition of the probabilities being equal. Suppose the state with energy $(1/2)\hbar\omega$ has a probability amplitude of $\cos\theta$. In this case, deduce $\langle E \rangle$, $\langle E^2 \rangle$, and ΔE. Now find $\langle x \rangle$ and $\langle x^2 \rangle$. With this result, choosing appropriate values for θ and t, show that you can reproduce the result for $\langle p \rangle$ in the previous part.

(c) Show that for a harmonic oscillator, the expectation value of position $\langle x \rangle$ satisfies the classical equation.

3 The Virial Theorem

Consider a one-dimensional system with the Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$, where $V(x) = \lambda x^n$.

(a) Calculate the commutator $\left[\hat{H}, \hat{x}\hat{p}\right]$.

(b) By taking the expectation value of this commutator, show that for any eigenstate of $\hat{H}$, one has the relation:

$$2\langle T \rangle = n\langle V \rangle,$$

where $\hat{T} \equiv \hat{p}^2/2m$ is the kinetic energy operator. Check this relation on the harmonic oscillator.

(c) Generalise this result to three dimensions by calculating $\left[\hat{H}, \hat{\mathbf{r}}\cdot\hat{\mathbf{p}}\right]$ and considering the potential $V(\mathbf{r})$ to be a **homogeneous function** of the variables x, y, and z, of degree n.

Note: A homogeneous function of degree n satisfies

$$V(\alpha x, \alpha y, \alpha z) = \alpha^n V(x, y, z)$$
$$\mathbf{r}\cdot\nabla V = nV$$

(d) Show that, for an arbitrary potential $V(r)$, one has the general relation

$$2\langle T\rangle = \left\langle r\frac{\partial V}{\partial r}\right\rangle.$$