DS 12: Angular Momentum and Symmetries

Philip Cherian

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1 Quantum Mechanical Vector Identities

(a) Begin by convincing yourself of the following identities:

$$[x_i, x_j] = 0$$
$$[p_i, p_j] = 0$$
$$[x_i, p_j] = i\hbar\delta_{ij}$$

(b) We define the dot product and the cross product as

$$\mathbf{a}\cdot\mathbf{b} = a_i b_i$$
$$(\mathbf{a}\times\mathbf{b})_k = \epsilon_{ijk} a_i b_j$$

Use this to show that

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} + [a_i, b_i]$$
$$(\mathbf{a}\times\mathbf{b})_k = -(\mathbf{b}\times\mathbf{a})_k + \epsilon_{ijk}[a_i, b_j]$$

(c) Using the above identities, show that

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = -\mathbf{p}\times\mathbf{r},$$
$$\mathbf{r}\cdot\mathbf{L} = 0 = \mathbf{p}\cdot\mathbf{L}.$$

(d) Show that

$$(\mathbf{a}\times\mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a}\cdot\mathbf{b})^2 - a_j[a_j, b_k]b_k + a_j[a_k, b_k]b_j - a_j[a_k, b_j]b_k - a_j a_k[b_k, b_j]$$

and verify that if $[a_i, b_j] = \gamma\delta_{ij}$ and $[b_i, b_j] = 0$

$$(\mathbf{a}\times\mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a}\cdot\mathbf{b})^2 + \gamma\mathbf{a}\cdot\mathbf{b}$$

and use this to show that

$$\mathbf{L}^2 = \mathbf{r}^2\mathbf{p}^2 - (\mathbf{r}\cdot\mathbf{p})^2 + i\hbar\mathbf{r}\cdot\mathbf{p}.$$

Properties of Angular Momentum

(a) Show that

$$[L_i, x_j] = i\hbar\epsilon_{ijk}x_k$$
$$[L_i, p_j] = i\hbar\epsilon_{ijk}p_k$$

(b) Use the above relations to show that

$$\begin{aligned}
\mathbf{p}\cdot(\mathbf{p}\times\mathbf{L}) &= 0\\
(\mathbf{p}\times\mathbf{L})\cdot\mathbf{p} &= 2i\hbar\mathbf{p}^2\\
(\mathbf{p}\times\mathbf{L})^2 &= \mathbf{p}^2\mathbf{L}^2\\
\mathbf{x}\cdot(\mathbf{p}\times\mathbf{L}) &= \mathbf{L}^2\\
(\mathbf{p}\times\mathbf{L})\cdot\mathbf{x} &= \mathbf{L}^2 + 2i\hbar(\mathbf{p}\cdot\mathbf{x})
\end{aligned}$$

The following identities of the Levi-Civita symbol will be useful:

$$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp},$$
$$\epsilon_{ijk}\epsilon_{ijq} = 2\delta_{kq}.$$